The constrained bridge index of links in the 3-sphere

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1 Introduction

An n-component *link* is the union of n mutually disjoint 1-spheres in the 3-sphere S^3. In particular, we call a 1-component link a *knot*. We say that a knot K is *trivial* if K bounds a disk in S^3. We take a height function $h : S^3 \to [0, 1]$, that is, h is a Morse function whose critical point set consists of two points, a maximum p_1 of height 1 and a minimum p_0 of height 0. Throughout this paper, we fix h. Let L be a link. Then $[L]$ denotes the ambient isotopy class of L. Then by slightly deforming L by ambient isotopy, if necessary, we may suppose that $h|_L : L \to [0, 1]$ is a Morse function. Then the *bridge number* of L, denoted by $b(L)$, is the number of maxima (= the number of minima) for $h|_L$. The *bridge index* of L, denoted by $b([L])$, is defined as follows;

$$b([L]) = \min\{b(L') \mid L' \in [L],\ h|_{L'} \text{ is a Morse function}\}.$$

It is easy to see that the bridge index of a knot is 1 if and only if the knot is a trivial knot. We say that L is in a *minimal bridge position* if L satisfies $b(L) = b([L])$.

The bridge index $b([L])$ of a link L was defined by H. Schubert [SH1] and has been one of the fundamental invariants in knot and link theory. For example, Schubert showed that the quantity (bridge index)-1 is additive for connected sum of knots [SH1]. Further Schubert studied 2-bridge position of 2-bridge knots, and showed that each 2-bridge knot admits unique 2-bridge position [SH2]. J. S. Birman showed that there exists a knot which admits two 3-bridge positions [B]. Y. Jang showed that there exists a 3-bridge knot which admits infinitely many different 3-bridge positions [J]. On the other hand, the concept of bridge index and bridge number have been generalized by many authors. For example, bridge number is refined as width of links, and by using the concept, the position called thin position was introduced by D. Gabai [Ga]. N. H. Kuiper defined what is called the superbridge index [Ku], and H. Goda defined the bridge index for spatial theta-curves [Go]. Particularly, H. Doll defined genus g bridge number by using Heegaard surface [D]. In this paper, we propose other new bridge indices for links called constrained bridge index.

In this paper, we mainly treat 2-component link $L = K_1 \cup K_2$ such that K_1 is a trivial knot. We introduce a new bridge index of L, denoted by $b_{K_1=1}([L])$, as follows.

$$b_{K_1=1}([L]) = \min\left\{ b(L') \;\middle|\; \begin{array}{l} L' = K_1' \cup K_2' \in [L],\ h|_{L'} \text{ is a Morse function} \\ \text{with } b(K_1') = 1, \text{ where } K_1' \text{ is the component} \\ \text{corresponding to } K_1 \end{array} \right\}.$$

In other words, it is the bridge index under the constraint $b(K_1) = 1$. We say that L is in a *minimal bridge position with respect to trivial* K_1 if L satisfies both $b(K_1) = 1$ and $b(L) = b_{K_1=1}([L])$. In general, the inequality $b_{K_1=1}([L]) \geq b([L])$ holds, and it is natural to ask whether there exist examples which make the inequalities strict. Then in Section 3, we show that for each integer $n(\geq 2)$ there exists a link $L_n = K_{1n} \cup K_{2n}$ satisfying $b_{K_{1n}=1}([L_n]) - b([L_n]) = n - 1$. Concretely speaking, for each n (≥ 2), let $L_n = K_{1n} \cup K_{2n}$ be the 2-component link such that K_{1n} is a trivial knot as in Figure 1, where K_{2n} is an $(n+1, n)$-torus knot. Then we have:

Proposition 1.1. For each n (≥ 2), let $L_n = K_{1n} \cup K_{2n}$ be the 2-component link such that K_{1n} is a trivial knot, and K_{2n} is an $(n+1, n)$-torus knot as in Figure 1. Then we have:

1. $b_{K_{1n}=1}([L_n]) = 1 + 2n$; and
2. $b([L_n]) = 2 + n$.

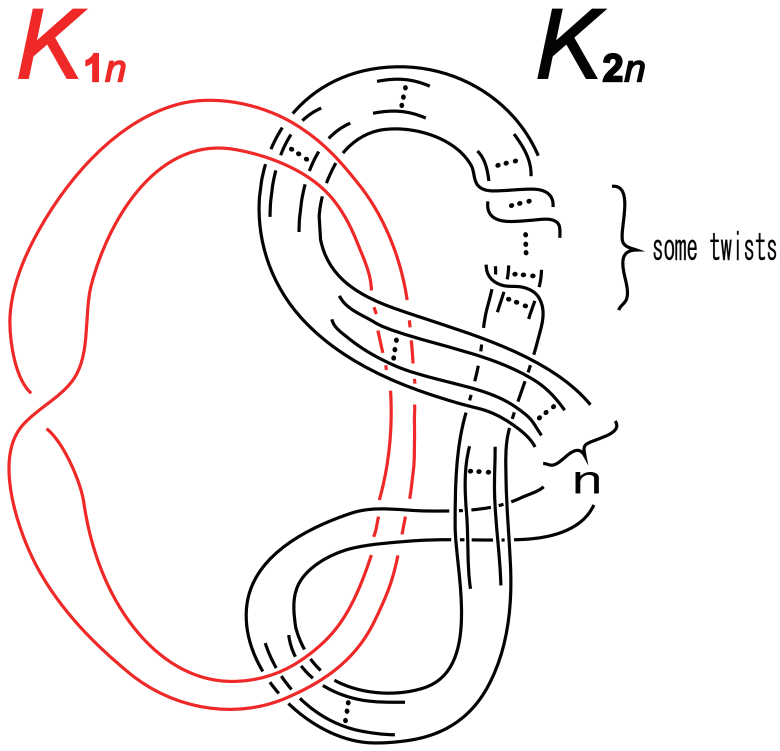


Figure 1: $L_n = K_{1n} \cup K_{2n}$

Then in Section 4 of this paper, we give generalizations of the concept of $b_{K_1=1}([L])$. In fact, we give a sequence of new bridge indices denoted by $b_{K_1=n}([L])$ ($n = 1, 2, \dots$) for 2-component link $L = K_1 \cup K_2$. For each integer n ($\geq b([K_1])$), we define a new bridge index called the *constrained bridge index* (of L) with respect to n-bridge K_1, denoted by $b_{K_1=n}([L])$, as follows;

$$b_{K_1=n}([L]) = \min \left\{ b(L') \;\middle|\; \begin{array}{l} L' = K_1' \cup K_2' \in [L],\ h|_{L'} \text{ is a Morse function} \\ \text{with } b(K_1') = n,\ \text{where } K_1' \text{ is the component} \\ \text{corresponding to } K_1 \end{array} \right\}.$$

In other words, it is the bridge index under the constraint $b(K_1) = n$. We note that $b_{K_1=1}([L])$ is the constrained bridge index with respect to 1-bridge K_1.

Remark 1.2. We can immediately generalize the constrained bridge index for links with ℓ (≥ 3) components $L = K_1 \cup \cdots \cup K_\ell$ as follows:

$$b_{K_1=n}([L]) = \min \left\{ b(L') \;\middle|\; \begin{array}{l} L' = K_1' \cup \cdots \cup K_\ell' \in [L],\ h|_{L'} \text{ is a Morse} \\ \text{function with } b(K_1') = n,\ \text{where } K_1' \text{ is the} \\ \text{component corresponding to } K_1 \end{array} \right\}.$$

We say that L is in a *minimal bridge position with respect to* n-bridge K_1 if L satisfies both $b(K_1) = n$ and $b(L) = b_{K_1=n}([L])$. Particularly, we consider the case when $b([K_1]) = 1$. We are interested in the sequence $\{b_{K_1=n}([L])\}_{n=1,2,\dots}$. We first note that for large n, the behavior of $\{b_{K_1=n}([L])\}$ is very simple. The precise statement is the following:

Proposition 1.3. Let $L = K_1 \cup K_2$ be a 2-component link. Let N be a positive integer defined as follows;

$$N = \min \left\{ b(K_1') \;\middle|\; \begin{array}{l} L' = K_1' \cup K_2' \in [L], h|_{L'} \text{ is a Morse function,} \\ \text{where } b(K_2') = b([K_2]) \end{array} \right\}.$$

Then, for each $n \geq N$, the following equality holds;

$$b_{K_1=n}([L]) = b([K_2]) + n.$$

According to Proposition 1.3, it is enough to consider $b_{K_1=n}([L])$ for $n < N$.

We show that there exist links $L = K_1 \cup K_2$ such that we can explicitly calculate the value $b_{K_1=n}([L])$ for each $n \geq 1$, which imply an interesting behavior of the sequence $\{b_{K_1=n}([L])\}_{n=1,2,\dots}$. The precise statement is as follows: Let m (≥ 4) be an integer, and $\alpha_1, \alpha_2, \dots, \alpha_{m-1}$ be integers such that $\alpha_j \neq -1, 0, \text{or } 1$ $(j = 1, 2, \ldots, m-1)$. Let $V_1 \subset V_2 \subset \cdots \subset V_m$ be a sequence of unknotted solid tori in S^3 such that, for $j = 1, 2, \ldots, m-1$, the core of V_j is parallel in V_{j+1} to a $(1, \alpha_j)$-curve (: a curve which goes around the boundary of V_{j+1} meridionally once, and longitudinally α_j times). Then we denote the core of V_j by K_j. Furthermore, we denote the closure of the exterior of V_i $(i = 1, 2, \ldots, m)$ by V_i^* (we note that each V_i^* is a solid torus), and the core of V_i^* by K_i^*. Let L denote the link $K_1 \cup K_m^*$. Then, for the constrained bridge index $b_{K_1=n}([L])$ with respect to n-bridge K_1, we have the next theorem.

Theorem 1.4. Let α_j $(j = 1, 2, \ldots, m-1), L = K_1 \cup K_m^*$ be as above. Then we have;

1. $b_{K_1=n}([L]) = n + |\prod_{j=2}^{m-1} \alpha_j|$ (if $1 \leq n < |\alpha_1|$),

2. $b_{K_1=n}([L]) = n + |\prod_{j=3}^{m-1} \alpha_j|$ (if $|\alpha_1| \leq n < |\alpha_1 \cdot \alpha_2|$),
   $b_{K_1=n}([L]) = n + |\prod_{j=4}^{m-1} \alpha_j|$ (if $|\alpha_1 \cdot \alpha_2| \leq n < |\alpha_1 \cdot \alpha_2 \cdot \alpha_3|$),

   $\vdots$

   $b_{K_1=n}([L]) = n + |\alpha_{m-2} \cdot \alpha_{m-1}|$ (if $|\prod_{j=1}^{m-4} \alpha_j| \leq n < |\prod_{j=1}^{m-3} \alpha_j|$),
   $b_{K_1=n}([L]) = n + |\alpha_{m-1}|$ (if $|\prod_{j=1}^{m-3} \alpha_j| \leq n < |\prod_{j=1}^{m-2} \alpha_j|$),

3. $b_{K_1=n}([L]) = n + 1$ (if $n \geq |\prod_{j=1}^{m-2} \alpha_j|$).

Remark 1.5. Note that $(1, \alpha_j)$-curve in ∂V_{j+1} is a $(1, \alpha_j)$-torus knot. By the classification for torus knot (Section 3. C of [R]), we see that $(1, \alpha_j)$-torus knot is a trivial knot.

Example 1.6. In the above, take $m = 5$, $\alpha_1 = 5$, $\alpha_2 = 4$, $\alpha_3 = 3$, $\alpha_4 = 2$. See Figure 2. Theorem 1.4 shows that the first 4 terms of the sequence $\{b_{K_1=n}([L])\}_{n=1,2,\ldots}$ are;

$$25(= 1 + 24),\ 26(= 2 + 24),\ 27(= 3 + 24),\ 28(= 4 + 24).$$

The terms $b_{K_1=5}([L]), b_{K_1=6}([L]), \ldots, b_{K_1=19}([L])$ are;

$$17(= 5 + 12),\ 18(= 6 + 12),\ \ldots,\ 31(= 19 + 12).$$

The terms $b_{K_1=20}([L]), b_{K_1=21}([L]), \ldots, b_{K_1=59}([L])$ are;

$$22(= 20 + 2),\ 23(= 21 + 2),\ \ldots,\ 61(= 59 + 2).$$

For $n \geq 60$, $b_{K_1=n}([L]) = n+1$.
The graph of $\{b_{K_1=n}([L])\}_{n=1,2,\ldots}$ of the example is as in Figure 3.

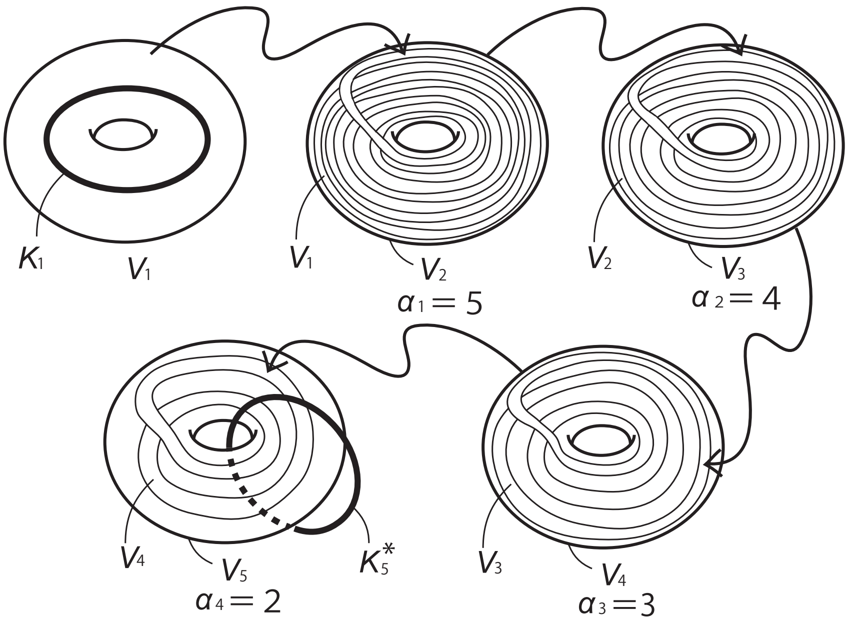


Figure 2: a figure for an example

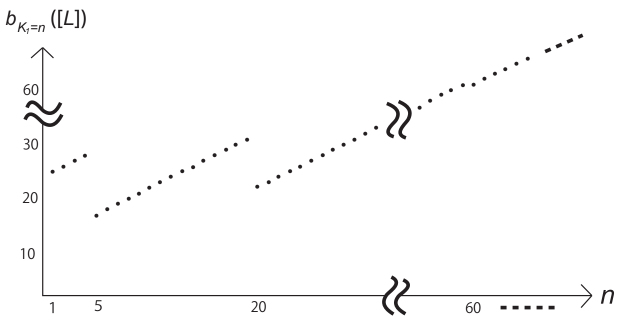


Figure 3: the behavior of $\{b_{K_1=n}([L])\}_{n=1,2,\ldots}$

This paper is organized as follows. Section 2 is the preliminaries. We give the definitions of fundamental concepts in 3-dimensional topology and link theory. Particularly we give the definitions of torus knot, satellite knot, and iterated torus knot which will be intensively used in Sections 3, and 4.

Further we introduce a theorem (Theorem 2.1) on the estimation of bridge index of satellite knot proved by Schubert ([SH1]).

In Section 3, we study the bridge index $b_{K_1=1}([L])$ in case when L is a satellite link. We introduce a new complexity on L, called dual index and use it to give an estimation of $b_{K_1=1}([L])$ (Theorem 3.1), which is an analogy of Theorem 2.1. Theorem 3.1 is proved by using the idea of the modern proof of Theorem 2.1 given by J. Schultens ([SJ]). Then by using Theorem 3.1, we prove Proposition 1.1. The results in Section 3 were already published in [K1].

In Section 4, we prove Theorem 1.4. The key of the proof is detailed analysis of taut essential tori in the exterior of L, that is accomplished by using generalizations of the arguments in [SJ] for more than one essential tori. The results in Section 4 were published in [K2].

In Section 5, we quickly review results in a paper of A. Zupan [Z]. In fact, in [Z], for a link L, a sequence of genus g bridge indices, called bridge spectrum, is introduced, and bridge spectrum of iterated torus knot is studied. We show that the sequence of constrained bridge indices and the bridge spectrum can be unified as the index denoted by $b_{K_1=n,g}([L])$. Further we show that $b_{K_1=n,g}([L])$ has another representative that uses the concept of Heegaard splitting.

2 Preliminaries

In this paper, we work in the differentiable category.

Let M be a compact orientable 3-manifold. We say that M is *irreducible* if each 2-sphere in M bounds a 3-ball in M. The 3-manifold M is called *reducible* if it is not irreducible. Let F be a surface properly embedded in M. Let s be a simple closed curve in F. We say that s is *inessential* if s bounds a disk in F, and s is essential if it is not inessential. We say that a disk D is a *compressing disk* for F if $D \cap F = \partial D$ and ∂D is an essential simple closed curve in F. We say that F is *compressible* if F has a compressing disk. Otherwise, F is *incompressible*. We say that M is *∂-irreducible* if ∂M is incompressible in M.

For a link L, $N(L)$ denotes a regular neighborhood of L. Then the exterior of L, denoted $E(L)$, is the closure of the exterior of $N(L)$. An essential simple closed curve in $\partial N(L)$ is called a *meridian* if it bounds a disk in $N(L)$, and an essential simple closed curve in $\partial N(L) = \partial E(L)$ is called a *longitude* if it represents a trivial element in $H_1(E(L))$. See Figures 4, and 5.

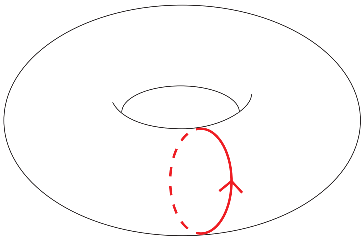


Figure 4: a meridian

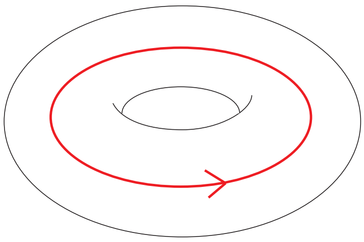


Figure 5: a longitude

A link L is called a *split link* if there exists a 2-sphere S^2 in the 3-sphere S^3 such that $S^2 \cap L = \emptyset$, and that S^2 separates components of L. Otherwise, L is a *non-split link*. Two links L and L' are called *ambient isotopic* if there is an ambient isotopoy of S^3 which sends L to L', i.e. there exists an isotopy $\varphi_t : S^3 \to S^3$ $(0 \leq t \leq 1)$ such that $\varphi_0 = \mathrm{id}$, and $\varphi_1(L) = L'$.

Let $\widehat{V}$ be an unknotted torus in S^3. Then let $T = \partial \widehat{V}$. For relatively prime integers p, q $(\neq 0)$, a knot K is a (p,q)-*torus knot* if K wraps around T in the meridional direction p times and in the longitudinal direction q times. See Figure 6 for example.

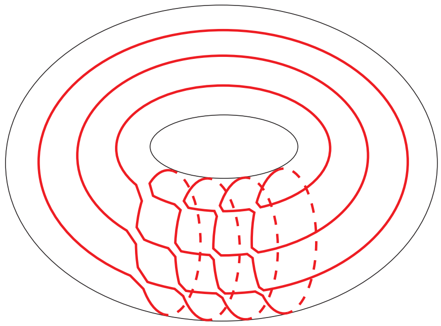


Figure 6: $(4,3)$-torus knot

Let L^0 be a non-trivial knot, and $\widetilde{V}$ be a small regular neighborhood of L^0. Let $\widehat{V}$ be an unknotted solid torus embedded in S^3, and K^0 be a knot in $\widehat{V}$, which is not ambient isotopic in $\widehat{V}$ to the core of $\widehat{V}$, and is not contained in a 3-ball in $\widehat{V}$. We fix a homeomorphism $\Psi : \widehat{V} \to \widetilde{V}$. Then $\Psi(K^0)$, which is denoted by K, is a knot in S^3. We say that K is a *satellite knot*. The image $\Psi(\widehat{V})$ is denoted by V. Now, we call L^0 a *companion* of K, V a *companion torus* of K with respect to L^0, and the pair $(\widehat{V}, K^0)$ the *pattern* of K with respect to L^0. Then, $\min\{\sharp(D \cap K^0) \mid D : \text{a meridian disk of } \widehat{V}\}$ is called the *index* of the pattern. See Figure 7 for example. For a bridge index of the satellite knot, Schubert gave the following:

Theorem 2.1. ([SH1], Satz 9) Let K be a satellite knot with L^0, and $(\widehat{V}, K^0)$ be as above. Let k be the index of $(\widehat{V}, K^0)$. Then the following inequality holds;

$$b([K]) \geq k \cdot b([L^0]).$$

We note that Schultens gave a modern proof of the above inequality in [SJ].

Remark 2.2. Recall that L^0 is a non-trivial knot. This assumption is essential in Theorem 2.1. In fact, for trivial L^0, we have a “satellite” knot K as in Figure 8. Here we note that $b([K]) = 2$, and $k = 3$. On the other hand, $k \cdot b([L^0]) = 3 \cdot 1 = 3$.

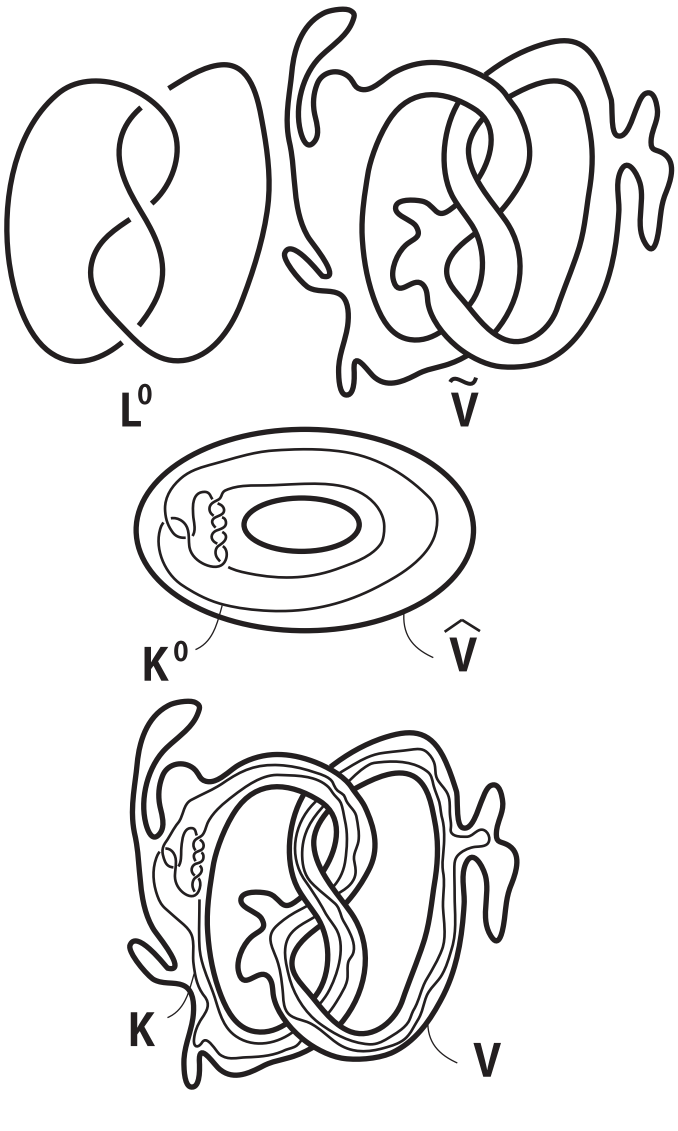


Figure 7: Satellite knot

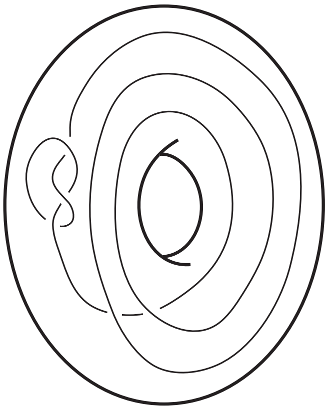


Figure 8: Example for the assumption of Theorem 2.1

Let K be a knot, and p, q $(\neq 0)$ be relatively prime integers. Then let $T_{p,q}$ $(\subset T = \partial\widehat{V})$ be a (p,q)-torus knot defined as above. Let $\widetilde{V}$ be a small regular neighborhood of K. We fix a meridian-longitude system on $\partial\widetilde{V}$. Further, let $\phi : \widehat{V} \to \widetilde{V}$ be a homeomorphism which sends the oriented meridian-longitude system to the oriented meridian-longitude system. Then the image $\phi(T_{p,q})$, denoted by $K_{p,q}$ is a knot in S^3. The knot $K_{p,q}$ is called a (p,q)-*cable* of K. See Figure 9 for example. Let $(p_0, q_0), (p_1, q_1), \ldots, (p_n, q_n)$

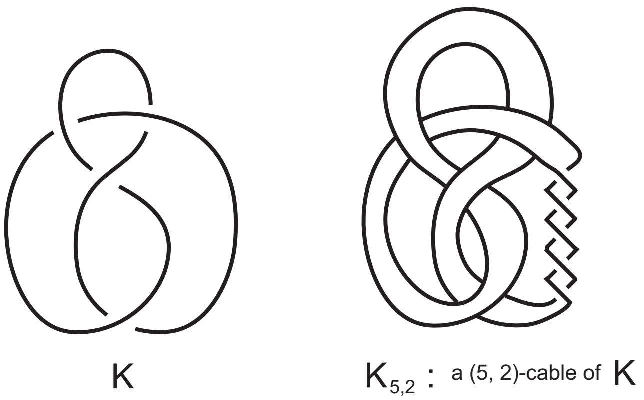


Figure 9: Cable knot

be a sequence of relatively prime integers. Then we define a sequence of cable knot $K_0, K_1, \ldots, K_n$ inductively as follows; $K_0 = (p_0, q_0)$-torus knot, and K_{i+1} is the (p_{i+1}, q_{i+1})-cable of K_i $(i = 0, 1, \ldots, n-1)$. Then we call K_n an *iterated torus knot* associated to $((p_0, q_0), (p_1, q_1), \ldots, (p_n, q_n))$.

3 A new bridge index for links with trivial knot components

Let $L = K_1 \cup K_2$ be a 2-component link such that K_1 is a trivial knot, and $b_{K_1=1}$ be the new bridge index of L introduced in Section 1. In this section, we show that for each n (≥ 2), there exists a link $L_n = K_{1n} \cup K_{2n}$ such that $b_{K_{1n}=1}([L_n]) - b([L_n]) = n - 1$. For demonstrating this, we give a result similar to Theorem 2.1, which works for satellite links (for the definition, see below). We firstly give a definition of satellite link, which is a generalization of satellite knot.

Let $L^0 = L_1^0 \cup \cdots \cup L_n^0$ ($n \geq 1$) be an n-component link in S^3 such that $E(L_1^0 \cup \cdots \cup L_n^0)$ is irreducible and ∂-irreducible, $\widetilde{V}_i$ ($i = 1, \ldots, n$) be a small regular neighborhood of L_i^0, and $\widehat{V}_i$ be an unknotted solid torus embedded in S^3. Let $K_i^0 (\subset \widehat{V}_i)$ be a knot which is not contained in a 3-ball in $\widehat{V}_i$ such that there exists $j \in \{1, \ldots, n\}$ such that K_j^0 is not ambient isotopic in $\widehat{V}_j$ to the core of $\widehat{V}_j$. We fix a homeomorphism $\Psi_i : \widehat{V}_i \to \widetilde{V}_i$ for each i. Then V_i denotes the image of $\widehat{V}_i$. Then T_i denotes ∂V_i, and we put $V = V_1 \cup \cdots \cup V_n$ and $T = T_1 \cup \cdots \cup T_n$. Furthermore, K_i denotes the image of K_i^0. Thus each K_i is a knot in S^3, and then L denotes the link $K_1 \cup \cdots \cup K_n$ in S^3. We call L a *satellite link*, L^0 a *companion* of L, and V a *companion tori* of L with respect to L^0. Moreover, we call the pair $(\widehat{V}_i, K_i^0)$ the *pattern* of K_i with respect to L_i^0. Then we call $\min\{\sharp(D_i \cap K_i^0) \mid D_i : \text{a meridian disk of } \widehat{V}_i\}$ the *index* of the pattern $(\widehat{V}_i, K_i^0)$.

Let $L = K_1 \cup K_2$ be a 2-component satellite link with a companion link $L^0 = L_1^0 \cup L_2^0$. Suppose that K_1 is a trivial knot. Then by Theorem 2.1, we can show that L_1^0 is a trivial knot. Let $N(L_1^0)$ be a small regular neighborhood of L_1^0. Since L_1^0 is a trivial knot, $E(L_1^0)$ is homeomorphic to a solid torus. We may regard L_2^0 as a knot in $E(L_1^0)$, hence the pair $(E(L_1^0), L_2^0)$ is a pattern. We denote the index of the pattern $(E(L_1^0), L_2^0)$ by k_1', and call it the *dual index* of L_1^0. With these terms, for constrained bridge index with respect to 1-bridge K_1, $b_{K_1=1}([L])$, we have the following theorem:

Theorem 3.1. Let $L = K_1 \cup K_2$ be a satellite link with a companion $L^0 = L_1^0 \cup L_2^0$ and a pattern $(\widehat{V}_i, K_i^0)$ ($i = 1, 2$) such that K_1 is a trivial knot. Let k_1' be the dual index of L_1^0, and k_i be the index of $(\widehat{V}_i, K_i^0)$. Suppose that K_1^0 is not ambient isotopic in $\widehat{V}_1$ to the core of $\widehat{V}_1$. Then the following inequality holds.

$$b_{K_1=1}([L]) \geq 1 + k_1' \cdot k_2 \ .$$

The proof of Theorem 3.1 is carried out by using the arguments of a paper of Schultens' [SJ], which gives a modern proof of Theorem 2.1. Particularly

Lemma 3.5 below is essential. For the statement of the lemma, we introduce some terms, which are used in [SJ].

Let K be a satellite knot, and V be a companion torus of K. Then T denotes ∂V. By slightly deforming T by ambient isotopy, if necessary, we may suppose that $h|_T : T \to [0,1]$ is a Morse function. Then $\mathcal{F}_T$ denotes the singular foliation on T induced by the levels of $h|_T$. Let σ be a singular leaf corresponding to a saddle singularity in $\mathcal{F}_T$. We call σ a *saddle* of $\mathcal{F}_T$. We note that σ has a representative as a wedge product $\sigma = s_1 \vee s_2$, where s_1 and s_2 are circles in T. If either s_1 or s_2 is inessential in T, we call σ an *inessential saddle*, and we call σ an *essential saddle* if it is not an inessential saddle. Let S_σ be the level sphere which contains σ. Then we can choose circles c_1, c_2 in T, which are parallel to s_1, s_2 respectively, in a certain level sphere S which is either slightly higher or slightly lower to S_σ. Now, $c_1 \cup c_2$ bounds an annulus on the level sphere S. Then σ is called a *nested saddle* if a small regular neighborhood of $c_1 \cup c_2$ in the annulus is contained in V(for example, it is as the left one in Figure 10). Otherwise, σ is a *non-nested saddle* (for example, it is as the right one in Figure 10). We say that T is *taut* with respect to $b([K])$ if the number of critical points of $h|_T$ is minimal in the isotopy class of T under the constraint that the knot which is ambient isotopic to K is in a minimal bridge position. Now, the following holds.

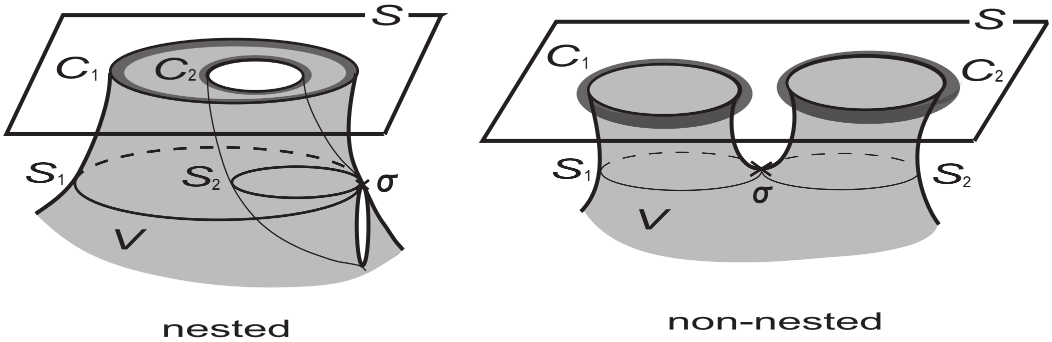


Figure 10: a nested saddle and a non-nested saddle

Lemma 3.2. ([SJ], LEMMA 1) Let K, V, T be as above. If $\mathcal{F}_T$ contains an inessential saddle, then there is an ambient isotopy ϕ_t $(0 \leq t \leq 1)$ of S^3 that does not change the number of critical points of K and T such that there exists an inessential saddle $\sigma^0 = s_1^0 \vee s_2^0$ of $\mathcal{F}_T$, where s_1^0 bounds a disk D_1 in $\phi_1(T)$ satisfying the following conditions:

1. The restriction of $\mathcal{F}_{\phi_1(T)}$ to D_1 consists of exactly one central singular point and concentric circles; and

2. There exists a disk component $\widetilde{D}_1$ in $S_{\sigma^0} \setminus s_1^0$ such that we can take a 3-ball B in S^3 bounded by $\widetilde{D}_1 \cup D_1$ such that B does not contain p_0 or p_1, where p_0 (p_1 resp.) is the minimum (maximum resp.) of h, and s_2^0 does not meet B.

Proof. The first condition on σ^0 is satisfied by choosing σ^0 to be an inessential saddle in $\mathcal{F}_T$ that is innermost in T and s_1^0 bounds a disk D_1. Then D_1 is either above or below the level sphere S_{σ^0}. Since the argument is symmetric, we may suppose that D_1 is above S_{σ^0}. Let $B^+_{\sigma^0}$ be the 3-ball in S^3 bounded by

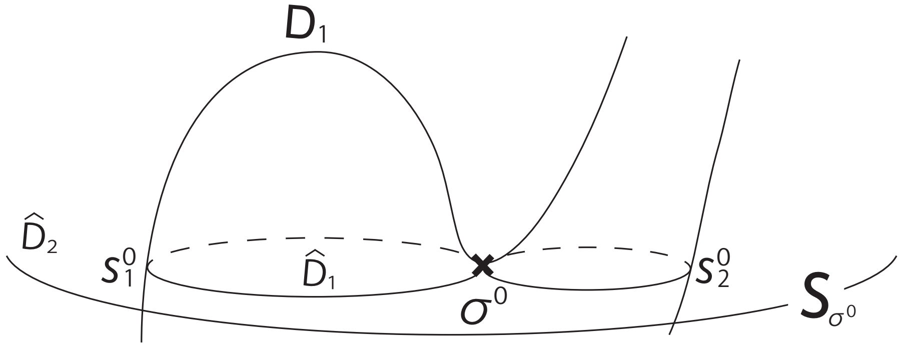


Figure 11: The case $s_2^0 \subset \widehat{D}_2$

S_{σ^0} such that $p_1 \in B^+_{\sigma^0}$. Then D_1 separates $B^+_{\sigma^0}$ into the 3-balls, say $\widehat{B}_1$ and $\widehat{B}_2$, where $p_1 \in \widehat{B}_2$. Then we denote the level disk $\widehat{B}_i \cap S_{\sigma^0}$ by $\widehat{D}_i$ $(i = 1, 2)$. If $s_2^0 \subset \widehat{D}_2$, we may take $B = \widehat{B}_1$, and $\widetilde{D}_1 = \widehat{D}_1$. See Figure 11. Thus, we suppose $s_2^0 \subset \widehat{D}_1$ (that is, it looks like as in Figure 12). We note that the critical point of D_1 is a maximum, say a_0. Then we note that there exists a monotonously increasing arc α disjoint from K, beginning at a_0 to p_1 such that $\alpha \cap T = \{a_0, \ldots, a_n\}$, where $a_0, \ldots, a_n$ are maximal points in T of $h|_T$. In fact, α is obtained in the following manner. First, we start to draw the monotonously increasing arc from a_0. Then if the arc meets T, then we extend the arc so that it goes slightly below T along a path in T from the intersection point to the closest maximum, say a_1, of T. Then it goes through T at a_1, then we further extend the arc monotonously increasingly, and repeat the above arguments for intersecting points $a_0, \ldots, a_n$. After a finite number of the above steps, the arc goes from a highest point a_n to p_1. Thus we obtain the arc α form a_0 to p_1. Let β_1 be the subarc between a_n and p_1, and let C_1' be a collar neighborhood of β_1. After a small isotopy, $T \cap C_1'$ consists of a small disk $D_1 = a_n \times (\text{a disk}) \subset T$. Let C_1^0 be a small 3-ball centered at p_1 that is disjoint from T. Set $C_1 = C_1' \cup C_1^0$ and consider

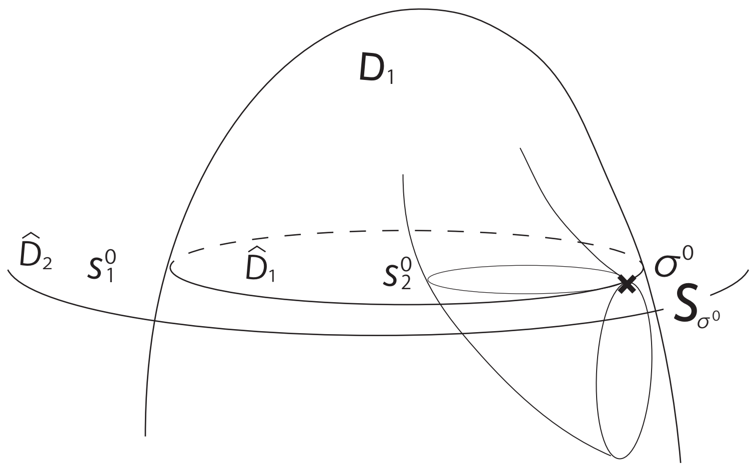


Figure 12: The case $s_2^0 \subset \widehat{D}_1$

$T_1' = (T \setminus D_1) \cup \mathrm{cl}(\partial C_1 \setminus D_1)$. Note that T_1' is ambient isotopic to T, where the ambient isotopy is supported in a small neighborhood of C_1. We note that no critical points have introduced for $h|_K$ and $h|_T$. Then let β_2 be the subarc between a_{n-1} and p_1, and let C_2' be a collar neighborhood of β_2. After a small isotopy, $T \cap C_2'$ consists of a small disk $D_2 = a_{n-1} \times (\text{a disk}) \subset T$. Let C_2^0 be a small 3-ball centered at p_1 that is disjoint from T_1'. Set $C_2 = C_2' \cup C_2^0$ and consider $T_2' = (T_1' \setminus D_2) \cup \mathrm{cl}(\partial C_2 \setminus D_2)$. Note that T_2' is ambient isotopic to T, where the ambient isotopy is supported in a small neighborhood of C_1. By repeating the same arguments, we obtain a torus T_{n+1}' which is ambient isotopic to T. See Figure 13. Note the 3-ball corresponding to $\widehat{B}_2$ does not contain p_1, and the 3-ball corresponding to $\widehat{B}_1$ contains p_1. This completes the proof. □

Lemma 3.3. ([SJ], LEMMA 2**)** Let K, V, T be as above. If T is taut with respect to $b([K])$, then there are no inessential saddles in $\mathcal{F}_T$

Proof. Suppose that there is an inessential saddle σ^0 satisfying the conclusions of Lemma 3.2. Without loss of generality, we may assume D_1 contains a maximum and lies above S_{σ^0}. Here $(K \cup T) \cap \mathrm{int}(B)$ can be shrunk horizontally and lowered via an isotopy to lie just below $\widetilde{D}_1$. This does not change the number of critical points of $h|_T$ and $h|_K$. After a small tilt, we can lower the number of critical points of $h|_T$. See Figure 14. This contradicts the assumption. □

Lemma 3.4. ([SJ], LEMMA 3**)** Let K, V, T be as above. If T is taut with respect to $b([K])$, then $\mathcal{F}_T$ has no nested saddles.

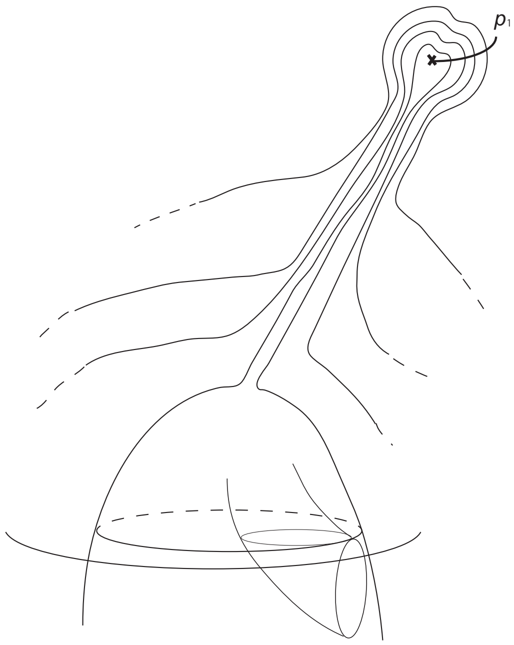


Figure 13: T'_{n+1}

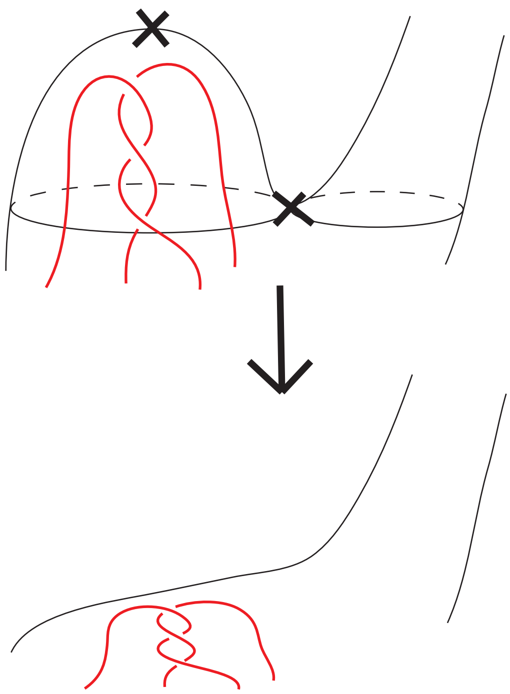


Figure 14: Reduce two critical points of $h|_T$

Proof. Let $\sigma' = s'_1 \vee s'_2$ be a highest saddle in $h|_T$. Let c'_1, c'_2 be circles in a level sphere S which is slightly lower than $S_{\sigma'}$ as above, and let $\widehat{D}'_1$, $\widehat{D}'_2$ be mutually disjoint disks bounded by c'_1, c'_2 in S respectively. Let c' be a component of $T \cap \mathrm{int}(\widehat{D}'_i)$ ($i = 1$ or 2). Then since σ' is the highest saddle, we see that c' bounds a disk $D_{c'}$ in T such that:

1. $D_{c'}$ is included in the region above S; and
2. The restriction of $\mathcal{F}_T$ to $D_{c'}$ consists of exactly one central singular point and concentric circles.

We push down the disk $D_{c'}$ slightly below S by an ambient isotopy as in Figure 15. We note that this isotopy can be performed so as not to change $b(K)$, and the number of critical points in $\mathcal{F}_T$. By repeating such isotopies, we may suppose that $\mathrm{int}(\widehat{D}'_i)$ is disjoint from T, i.e. $\widehat{D}'_i$ is contained in V or $\mathrm{cl}(S^3 \setminus V)$. Then since s'_i is essential in T, we see that c'_i is essential in T by the definition of c'_i. We note that since K is knotted, T is incompressible in $\mathrm{cl}(S^3 \setminus V)$. Hence the disk $\widehat{D}'_i$ must be a meridian disk in V. This shows that σ' is non-nested. Then if there exists a nested saddle in T, we see that there is a pair of a nested saddle and a non-nested saddle in T. In this situation, there exists an adjacent pair of saddles $\sigma_1 = s^1_1 \vee s^1_2$, $\sigma_2 = s^2_1 \vee s^2_2$ in $\mathcal{F}_T$ contained in T such that σ_1 is nested and σ_2 is non-nested. Then we note that there exists a component, say C, of $T \setminus (\sigma_1 \cup \sigma_2)$ which does not contain critical points. Without loss of generality, we suppose s^1_1 and s^2_1 meet C, and σ_1 lies above σ_2. Then we note that the component $T \setminus \sigma_1$, which is lying above σ_1 and is meeting both s^1_1 and s^1_2, is the open disk called D^1_3. Let D'_1 be the disk in S_{σ_1} bounded by s^1_2 such that D'_1 does not intersect s^1_1. Let $D = C \cup D^1_3 \cup D'_1$. It is easy to see that D is a disk such that $\partial D = s^2_1$. Then by using the argument which is similar to that of the proof of Lemma 3.3, we are able to lower D to lie just below $\widehat{D}_1$. After small tilt, we can remove σ_2. See Figure 16. It contradicts the assumption that T is taut with respect to $b([K])$. Thus we have that each saddle in T is non-nested. □

From Lemmata 3.3 and 3.4, we obtain the next lemma.

Lemma 3.5. ([SJ], Remark 2) Let K, V, T be as above. If T is taut with respect to $b([K])$, then each saddle in $\mathcal{F}_T$ is essential and non-nested.

Then we generalize Lemma 3.5 for links. Let $L = K_1 \cup \cdots \cup K_n$ ($n \geq 1$) be a satellite link with a companion $L^0 = L^0_1 \cup \cdots \cup L^0_n$. Let V_i, V, T_i, and T be as in the paragraph preceding Theorem 3.1. We suppose that $h|_T : T \to [0,1]$ is a Morse function. Then $\mathcal{F}_T$ denotes the singular foliation on T induced by the levels of $h|_T$, and we define a *saddle* σ as in the previous

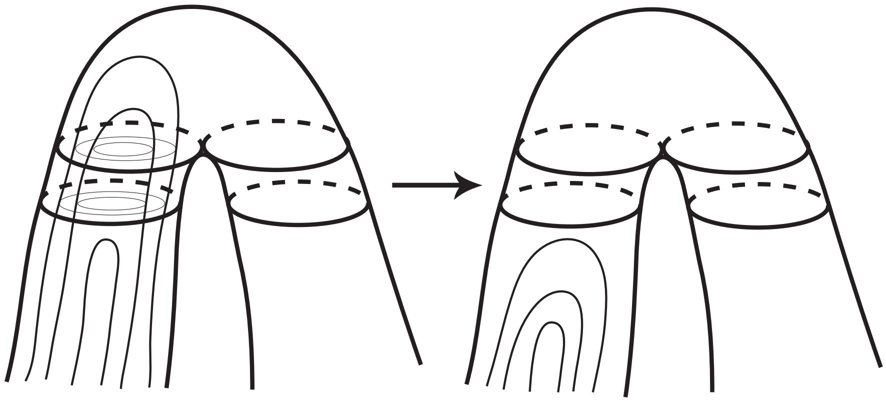


Figure 15: Push down the disk $D_{c'}$

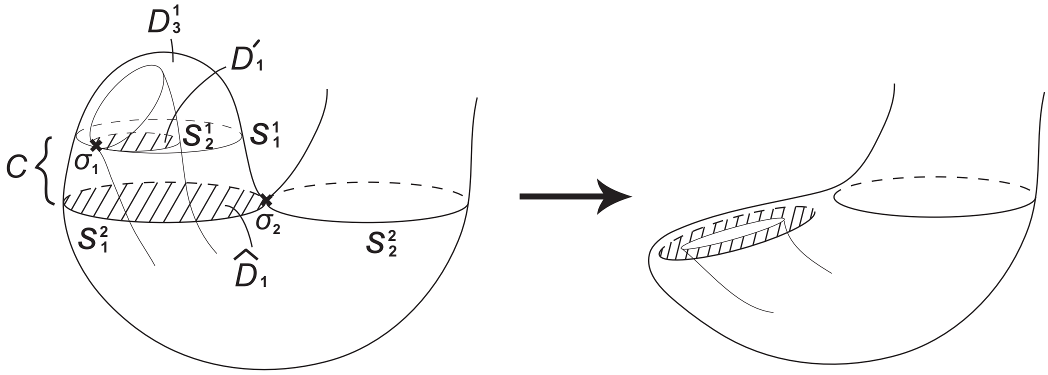


Figure 16: Remove σ_2

setting. Furthermore, we also define the terms *taut with respect to* $b([L])$, *nested saddles*, etc. as for satellite knot. Then, we introduce some lemmata and prove Theorem 3.1. Let σ be a saddle of $\mathcal{F}_T$. Recall that σ is a wedge product of two circles s_1, s_2 in T. Then as above, S_σ denotes the level sphere containing σ.

Lemma 3.6. (corresponding to Lemma 3.2) If $\mathcal{F}_T$ contains an inessential saddle, then there exists an ambient isotopy ϕ_t $(0 \leq t \leq 1)$ in S^3 that satisfies the following conditions:

1. The height function $h|_{\phi_1(L)}$ is a Morse function on $\phi_1(L)$, thus $b(\phi_1(L))$ is defined, and $h|_{\phi_1(T)}$ is a Morse function on $\phi_1(T)$, thus $\mathcal{F}_{\phi_1(T)}$ is defined;

2. We have $b(\phi_1(K_i)) = b(K_i)$ $(i = 1, \ldots, n)$, and the number of critical points of $h|_{\phi_1(T_i)}$ equals that of $h|_{T_i}$; and

3. There exists an inessential saddle $\sigma^0 = s_1^0 \vee s_2^0$ of $\mathcal{F}_{\phi_1(T)}$, where s_1^0 bounds a disk D_1 in $\phi_1(T)$ satisfying the following conditions:

   (a) The restriction of $\mathcal{F}_{\phi_1(T)}$ to D_1 consists of exactly one central singular point and concentric circles; and

   (b) There exists a disk component $\widetilde{D}_1$ in $S_\sigma \setminus s_1^0$ such that we can take a 3-ball B in S^3 bounded by $\widetilde{D}_1 \cup D_1$ such that B does not contain p_0 or p_1, where p_0 (p_1 resp.) is the minimum (maximum resp.) of h, and s_2^0 does not meet B.

The proof of the above lemma is carried out by applying the arguments in the proof of Lemma 3.2, where knots are treated. We note that arguments completely work for the setting of Lemma 3.6

In the remainder of this section, we will restrict our attention to non-split 2-component satellite links such that one component of each link is a trivial knot. Let $L = K_1 \cup K_2$ be such a link with K_1 a trivial knot, $L^0 = L_1^0 \cup L_2^0$ be a companion of L, and $(\widehat{V}_i, K_i^0)$ $(i = 1, 2)$ be a pattern of K_i with respect to L_i^0. We use notations $V = V_1 \cup V_2$, $T = T_1 \cup T_2$, $\mathcal{F}_T$, k_1, k_1', k_2 etc. in the previous setting. We suppose that L is in a minimal bridge position with respect to trivial K_1. We say that T is *taut with respect to trivial* K_1 if the number of critical points of $h|_T$ is minimal in the isotopy class under the constraint that the link which is ambient isotopic to L is in a minimal bridge position with respect to trivial K_1. It is easy to prove the next lemma by using the arguments in the proof of Lemma 3.3, and we omit giving the proof here.

Lemma 3.7. (corresponding to Lemma 3.3) If T is taut with respect to trivial K_1, then there are no inessential saddles in $\mathcal{F}_T$.

Let σ_1, σ_2 be saddles of $\mathcal{F}_T$. We say that the pair σ_1, σ_2 is *adjacent* if there exists a component of $T \setminus (\sigma_1 \cup \sigma_2)$, which is denoted by C, such that there is no critical point of $h|_T$ in C. This term will be used in the proof of Lemma 3.8.

Lemma 3.8. (corresponding to Lemma 3.4) Suppose T is taut with respect to trivial K_1. If K_1^0 is not a core of $\widehat{V}_1$, then each saddle of $\mathcal{F}_T$ contained in T_1 is nested, and each saddle of $\mathcal{F}_T$ contained in T_2 is non-nested.

Proof. We first note that index k_1 is greater than 1, since K_1 is a trivial knot, and K_1^0 is not a core of $\widehat{V}_1$. By Lemma 3.7, each saddle in $\mathcal{F}_T$ is essential. Let $\sigma = s_1 \vee s_2$ be the highest saddle in $\mathcal{F}_T$. Then for σ, the next claim holds:

Claim 3.9. The saddle σ is non-nested.

Proof of Claim 3.9. The following arguments are essentially the same as the first half of the proof of Lemma 3.4. Let c_1, c_2 be circles in a level sphere S which is slightly lower than S_σ, and let $\widehat{D}_1$, $\widehat{D}_2$ be mutually disjoint disks bounded by c_1, c_2 respectively in S. Let c be a component of $T \cap \mathrm{int}(\widehat{D}_i)$ ($i =$ 1 or 2). Then since σ is the highest saddle, we see that c bounds a disk D_c in T such that:

1. D_c is included in the region above S; and
2. The restriction of $\mathcal{F}_T$ to D_c consists of exactly one central singular point and concentric circles.

We push down the disk D_c slightly below S by an ambient isotopy as in Figure 15. We note that this isotopy can be performed so as not to change $b(K_i)$ ($i = 1, 2$), and the number of critical points in $\mathcal{F}_T$. By repeating such isotopies, we may suppose that $\mathrm{int}(\widehat{D}_i)$ is disjoint from T, i.e. $\widehat{D}_i$ is contained in V or $\mathrm{cl}(S^3 \setminus V)$. Then since s_i is essential in T, we see that c_i is essential in T by the definition of c_i. We note that since L is a non-split link, L^0 is a non-split link. This implies that T is incompressible in $\mathrm{cl}(S^3 \setminus V)$. Hence the disk $\widehat{D}_i$ must be a meridian disk in V. This shows that σ is non-nested. □

Then, let $\sigma' = s_1' \vee s_2'$ be the saddle which is the highest one in the saddles of $\mathcal{F}_T$ contained in T_1. Then we have:

Claim 3.10. The saddle σ' is nested, in particular the saddle σ in Claim 3.9 is contained in T_2.

Proof of Claim 3.10. We take a level sphere S', circles c_1', c_2' ($\subset S'$), and disks $\widehat{D}_1'$, $\widehat{D}_2'$ analogous to $S, c_1, c_2, \widehat{D}_1, \widehat{D}_2$ for σ in the proof of Claim 3.9. Assume that σ' is non-nested. Then, the neighborhood of $\partial\widehat{D}_i'$ in $\widehat{D}_i'$ ($i = 1, 2$) is contained in V_1. Hence any component of $\mathrm{int}(\widehat{D}_i') \cap T$ which is outermost in $\mathrm{int}(\widehat{D}_i')$ is contained in T_1. Then the arguments in the proof of Claim 3.9 work and we may suppose that each $\widehat{D}_i'$, is contained in V_1. Since $k_1 > 1$, we see that K_1 intersects $\widehat{D}_i'$ ($i = 1, 2$) in at least 2 points. This shows that $b(K_1) > 1$, a contradiction. Hence σ' is nested. This together with Claim 3.9 shows that σ is contained in T_2. □

Then we have:

Claim 3.11. Each saddle in T_2 is non-nested.

Proof of Claim 3.11. If there exists a nested saddle in T_2, then by Claims 3.9, and 3.10, we see that there is a pair of a nested saddle and a non-nested saddle in T_2. In this situation, there exists an adjacent pair of saddles σ_1, σ_2 in $\mathcal{F}_T$ contained in T_2 such that σ_1 is nested and σ_2 is non-nested. Then by the same argument as in the proof of Lemma 3.4, we can derive a contradiction to the assumption that T is taut with respect to trivial K_1. See Figure 16. Thus we have that each saddle in T_2 is non-nested. □

Finally, we show the next:

Claim 3.12. Each saddle in T_1 is nested.

Proof of Claim 3.12. If there exists a non-nested saddle in T_1, then by Claim 3.10, we see that there is a pair of a nested saddle and a non-nested saddle in T_1. By the arguments in the proof of Claim 3.11, we can derive a contradiction. Thus we see that any saddle in T_1 is a nested saddle. □

Claims 3.11, and 3.12 complete the proof of Lemma 3.8. □

By using the above arguments, now we prove Theorem 3.1.

Proof of Theorem 3.1. Recall that L_1^0 is a trivial knot, hence the extorior of V_1, say V_1^c, is an unknotted solid torus. By Lemma 3.8, each saddle of T_1 ($= \partial V_1 = \partial V_1^c$) is essential and nested. Then let $\sigma' = s_1' \vee s_2'$ be the saddle which is the highest one in the saddles of T_1 and $\widehat{D}_i'$ ($i = 1, 2$) be the disk bounded by c_i' in S' as in the proof of Claim 3.10 in the proof of Lemma 3.8. We consider about $\widehat{D}_i' \cap T_2$. If there exists a component, say c, of $\widehat{D}_i' \cap T_2$ such that c is inessential in T_2, then by Lemma 3.7, there exists a disk D_c in T_2 such that $\partial D_c = c$ and the restriction of $\mathcal{F}_T$ to D_c consists of one central singularity and concentric circles. We note that D_c might be under S' (as in

Figure 17). By using D_c, we can apply an isotopy as in the proof of Claim 3.9 in the proof of Lemma 3.8 to remove c from $\widehat{D}'_i \cap T_2$. Hence, we may suppose that any component of $\widehat{D}'_i \cap T_2$ is essential in T_2. Thus by the definition of the dual index k'_1 of L^0_1, $\widehat{D}'_i \cap V_2$ consists of at least k'_1 meridian disks of V_2. Furthermore by the definition of k_2, K_2 intersects each meridian disk of V_2 at least k_2 times. This shows that K_2 intersects $\widehat{D}'_i$ at least $k'_1 \cdot k_2$ times, and this implies that K_2 has at least $k'_1 \cdot k_2$ maxima. This together with the fact $b(K_1) = 1$ gives the conclusion of Theorem 3.1. □

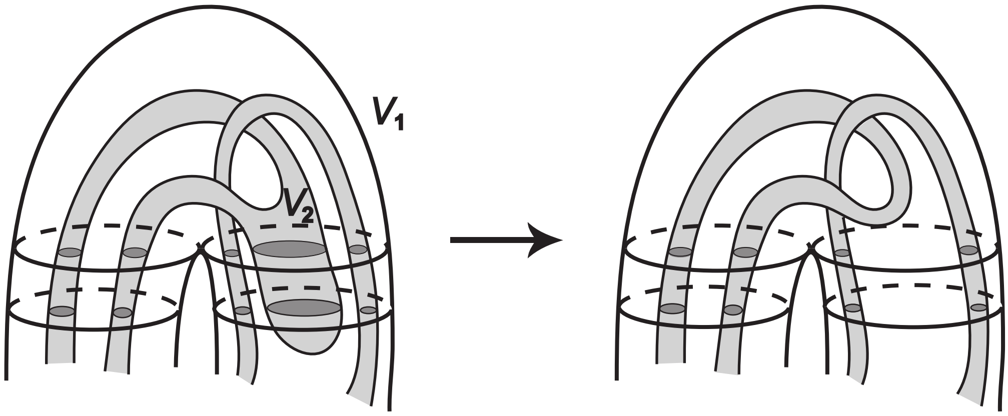


Figure 17: Remove c

Let $L = K_1 \cup K_2$ be a non-split 2-component link such that K_1 is a trivial knot. In general, $b([L]) \leq b_{K_1=1}([L])$ holds. Thus we would like to ask whether there exists L such that $b([L]) < b_{K_1=1}([L])$ holds. In fact, we prove the following.

Proposition 1.1. For each n (≥ 2), let $L_n = K_{1n} \cup K_{2n}$ be the 2-component link such that K_{1n} is a trivial knot, and K_{2n} is an $(n+1, n)$-torus knot as in Figure 18. Then we have:

1. $b_{K_{1n}=1}([L_n]) = 1 + 2n$; and
2. $b([L_n]) = 2 + n$.

Proof. Note that $L_n = K_{1n} \cup K_{2n}$ is a satellite link with the companion $L^0 = L^0_1 \cup L^0_2$ as in Figure 19-(a) and the pattern $(\widehat{V}_i, K^0_i)$ $(i = 1, 2)$ as in Figure 19 -(c). Then V denotes the companion torus $V_1 \cup V_2$. Further we let $T_i = \partial V_i$ $(i = 1, 2)$, and $T = T_1 \cup T_2$. (Figure 19-(b)). Firstly, we note that

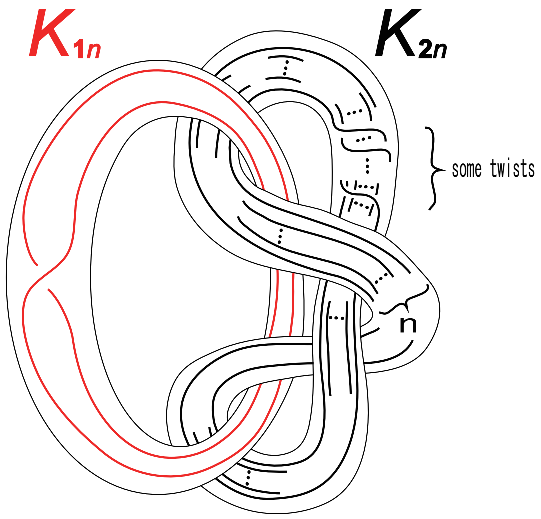


Figure 18:

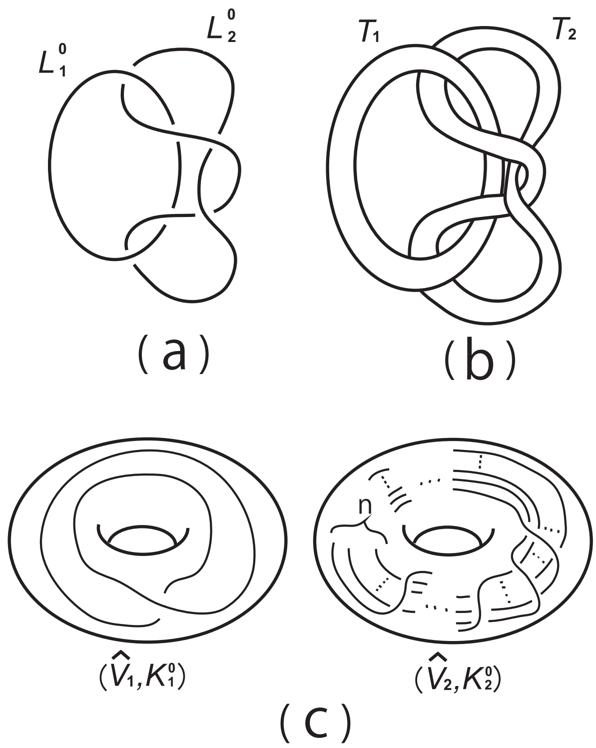


Figure 19:

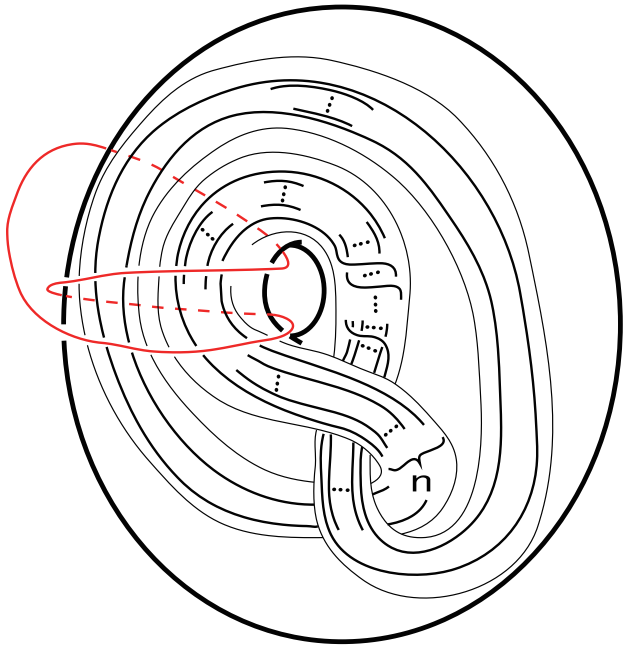


Figure 20:

the dual index of L_1^0 is 2, and the index of the pattern $(\widehat{V}_2, K_2^0)$ is n. Hence by Theorem 3.1, we have $b_{K_{1n}=1}([L_n]) \geq 1 + 2n$. Note that L_n can be isotoped into a position as in Figure 20, hence we see that $b_{K_{1n}=1}([L_n]) \leq 1 + 2n$. Thus we obtain $b_{K_{1n}=1}([L_n]) = 1 + 2n$. Next, by the facts $b([K_{1n}]) = 1$ and $b([K_{2n}]) = n$ ([Mu], Theorem 7.5.3), we have $b([L_n]) \geq 1 + n$. Assume that $b([L_n]) = 1 + n$. Let $L'_n = K'_{1n} \cup K'_{2n}$ ($\in [L_n]$) be a position such that $b(L'_n) = 1 + n$. This together with the facts, $b([K'_{1n}]) = 1$, and $b([K'_{2n}]) = n$ shows that $b(K'_{1n}) = 1$ (, and $b(K'_{2n}) = n$). This shows that $b_{K_{1n}=1}([L_n]) \leq 1 + n$, but this contradicts the above. Therefore we have $b([L_n]) \geq 2 + n$. On the other hand, by Figure 18, we see $b([L_n]) \leq 2 + n$. Thus we obtain $b([L_n]) = 2 + n$. □

4 The constrained bridge index

In this section, we prove Proposition 1.3 and Theorem 1.4.

Proof of Proposition 1.3

First, we prove the next proposition stated in Section 1.

Proposition 1.3. Let $L = K_1 \cup K_2$ be a 2-component link. Let N be a positive integer defined as follows;

$$N = \min \left\{ b(K_1') \;\middle|\; \begin{array}{l} L' = K_1' \cup K_2' \in [L], h|_{L'} \text{ is a Morse function,} \\ \text{where } b(K_2') = b([K_2]) \end{array} \right\}.$$

Then, for each $n \geq N$, the following equality holds;

$$b_{K_1=n}([L]) = b([K_2]) + n.$$

Proof. It is clear that, for each $n \geq b([K_1])$, we have $b_{K_1=n}([L]) \geq b([K_2])+n$. By the definition of N, there is $K_1' \cup K_2' \in [L]$ such that K_1' corresponds to K_1, $b(K_1') = N$, and $b(K_2') = b([K_2])$. Then for $n \geq N$, let $K_1'' \cup K_2'$ be a link obtained from $K_1' \cup K_2'$ by adding $n - N$ curls to K_1' locally as in Figure 21. Then by considering the number of maxima of $K_1'' \cup K_2'$, we obtain $b_{K_1=n}([L]) \leq b([K_2]) + n$, establishing the equality of the proposition. □

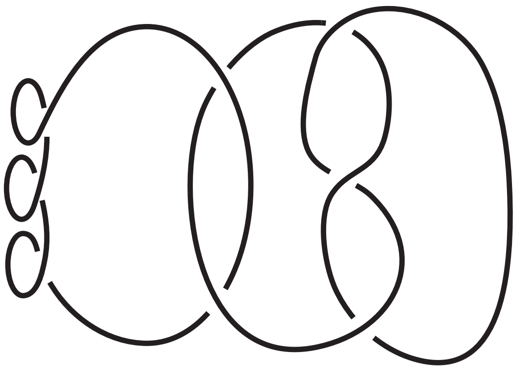


Figure 21: $K_1'' \cup K_2'$

Proof of Theorem 1.4

First, we recall Theorem 1.4. Let m (≥ 4) be an integer, and $\alpha_1, \alpha_2, \ldots, \alpha_{m-1}$ be integers such that $\alpha_j \neq -1, 0, \text{or } 1$ $(j = 1, 2, \ldots, m-1)$. Let $V_1 \subset V_2 \subset \cdots \subset V_m$ be a sequence of unknotted solid tori in S^3 such that, for $j = 1, 2, \ldots, m-1$, the core of V_j is parallel in V_{j+1} to a $(1, \alpha_j)$-curve (: a curve which goes around the boundary of V_{j+1} meridionally once, and longitudinally α_j times). Then we denote the core of V_j by K_j. Furthermore, we denote the closure of the exterior of V_i $(i = 1, 2, \ldots, m)$ by V_i^* (we note that each V_i^* is a solid torus), and denote the core of V_i^* by K_i^*. Let L denote the link $K_1 \cup K_m^*$. Let p, q be a pair of integers such that $1 \leq p < q \leq m$. Then $k(V_q, K_p)$ denotes $\min\{\sharp(D \cap K_p) \mid D : \text{ a meridian disk of } V_q\}$. Similarly, we denote $\min\{\sharp(D \cap K_q^*) \mid D : \text{ a meridian disk of } V_p^*\}$ by $k(V_p^*, K_q^*)$. Then the next holds.

Assertion 4.1. Let $k(V_q, K_p), k(V_p^*, K_q^*)$ be as above. Then the following equality holds;

$$k(V_q, K_p) = \left| \prod_{j=p}^{q-1} \alpha_j \right| = k(V_p^*, K_q^*).$$

Proof. By the definition, we see that any meridian disks of V_q $(q = 2, \ldots, m)$ intersects K_{q-1} in at least $|\alpha_{q-1}|$ points, and there exists a meridian disk of V_q, called D_q, which intersects K_{q-1} exactly in $|\alpha_{q-1}|$ points. By cut and paste arguments of 3-dimensional topology, we suppose that $V_{q-1} \cap D_q$ consists of $|\alpha_{q-1}|$ meridian disks of V_{q-1} (Figure 22). These show that $k(V_q, K_{q-2}) \geq |\alpha_{q-1} \cdot \alpha_{q-2}|$. On the other hand, it is easy to observe that there is a meridian disk of V_q which intersects k_{q-2} in $|\alpha_{q-1} \cdot \alpha_{q-2}|$ points. Hence we have $k(V_q, K_{q-2}) = |\alpha_{q-1} \cdot \alpha_{q-2}|$. By repeating similar arguments, for each integer i $(1 \leq i \leq q-p)$, we obtain $k(V_q, K_{q-i}) = |\alpha_{q-1} \cdots \alpha_{q-i}|$. For $k(V_p^*, V_q^*)$, the arguments similar to the above holds. □

In the following, T_i $(i = 1, 2, \ldots, m)$ denotes the boundary of V_i. We denote $T_1 \cup T_2 \cup \cdots \cup T_m$ by T. Recall, from Section 1, that $h : S^3 \to [0, 1]$ is a Morse function. We suppose that $h|_T : T \to [0, 1]$ is a Morse function. Then $\mathcal{F}_T$ denotes the singular foliation on T induced by the levels of $h|_T$. Let σ be the singular leaf corresponding to a saddle singularity in $\mathcal{F}_T$. We call σ a *saddle* of $\mathcal{F}_T$, as in Section 3. Then we can define inessential saddle, essential saddle as in Section 3, and we do not repeat to state the definitions again. Further we use S_σ to denote the level sphere containing σ, as in Section 3. We can also define nested, and non-nested saddle. For these concepts, since each T_i bounds a solid torus on both sides, the definitions are slightly subtle, hence, we will state the definitions.

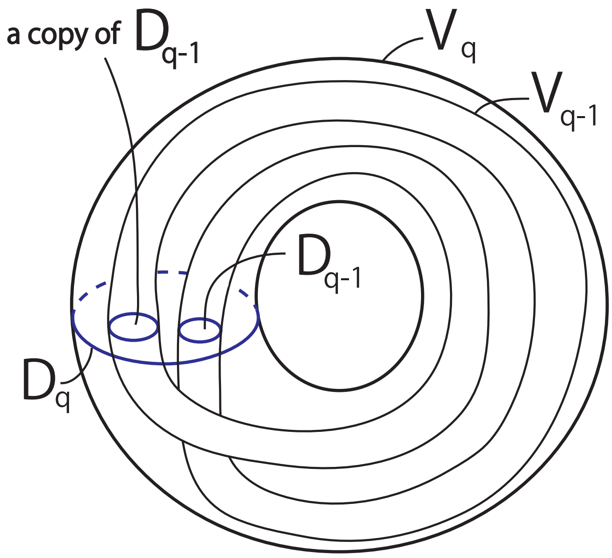


Figure 22: D_q intersects D_{q-1} and the copy of D_{q-1}

Let T_i be the component of T which contains σ. Then we can choose circles c_1, c_2 in T_i, which are parallel to s_1, s_2 respectively, in a certain level sphere S_σ^ε which is either slightly higher or slightly lower to S_σ. Now, $c_1 \cup c_2$ bounds an annulus on the level sphere S_σ^ε. Then σ is called a *nested saddle* if a small regular neighborhood of $c_1 \cup c_2$ in the annulus is contained in V_i. Otherwise, σ is a *non-nested saddle*.

Recall that L denotes the link $K_1 \cup K_m^*$. Then we note that we can define $b_{K_1=n}([L])$ for each $n \geq 1$, since $b([K_1]) = 1$. We say that T is *taut with respect to* n-bridge K_1, if the number of critical points of $h|_T$ is minimal in the ambient isotopy class of $L \cup T$ under the constraint that the link which is ambient isotopic to L is in a minimal bridge position with respect to n-bridge K_1.

We note that we can prove the next lemma by using the arguments as in Lemma 3.5, hence we omit the proof. (Note that the deformation used in the proof of Lemma 3.3 preserves the property "taut with respect to n-bridge K_1".)

Lemma 4.2. Let L, T be as above. If T is taut with respect to n-bridge K_1, then each saddle in $\mathcal{F}_T$ is essential.

By using similar arguments as in the proof of Lemma 3.8 of Section 3, we can prove the next lemma, and the proof is omitted;

Lemma 4.3. If T is taut with respect to n-bridge K_1, then for each T_i, all of the saddles of T_i are nested, or are non-nested.

Remark 4.4. In general, let V be an unknotted solid torus in S^3. Suppose each saddle of ∂V is essential and non-nested, then V looks like a small regular neighborhood of a trivial knot, that is, V admits a knee-thigh decomposition as in Figure 23. On the other hand, suppose each saddle of ∂V is essential and nested, then the closure of the exterior of V looks like a small regular neighborhood of a trivial knot. Hence Lemma 4.3 shows that for each i, V_i or V_i^* looks like a small regular neighborhood of a trivial knot.

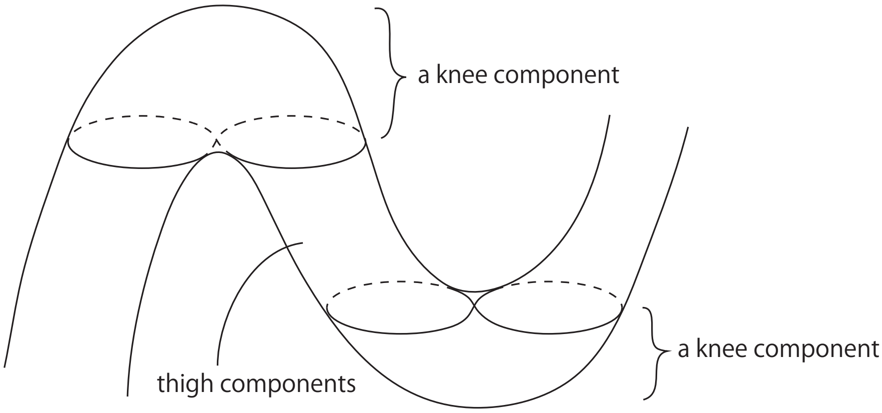


Figure 23: Knee-thigh decomposition

For each q $(= 2, \ldots, m-2)$, by Assertion 4.1, we obtain $k(V_q, K_1) = |\prod_{j=1}^{q-1} \alpha_j|$ and $k(V_{q+1}^*, K_m^*) = |\prod_{j=q+1}^{m-1} \alpha_j|$. These values together with Figure 24 give a proof of the next lemma.

Lemma 4.5. For each q $(= 2, \ldots, m-2)$, we have the following: for each n with $n \geq |\prod_{j=1}^{q-1} \alpha_j|$ we have; there exists a position of $L = K_1 \cup K_m^*$ such that $b(K_1) = n$ and $b(K_m^*) = |\prod_{j=q+1}^{m-1} \alpha_j|$. In particular, we have $b_{K_1=n}([L]) \leq n + |\prod_{j=q+1}^{m-1} \alpha_j|$.

We note that in Figure 24, each saddle of T_{q+1} is essential and nested. The next lemma shows this phenomena holds if T is taut with respect to n-bridge K_1, where $n < |\prod_{j=1}^{q} \alpha_j|$.

Lemma 4.6. For each q $(= 2, \ldots, m-2)$, we have the following: for each n with $n < |\prod_{j=1}^{q} \alpha_j|$ we have; if T is taut with respect to n-bridge K_1, then, each saddle of T_{q+1} is essential and nested.

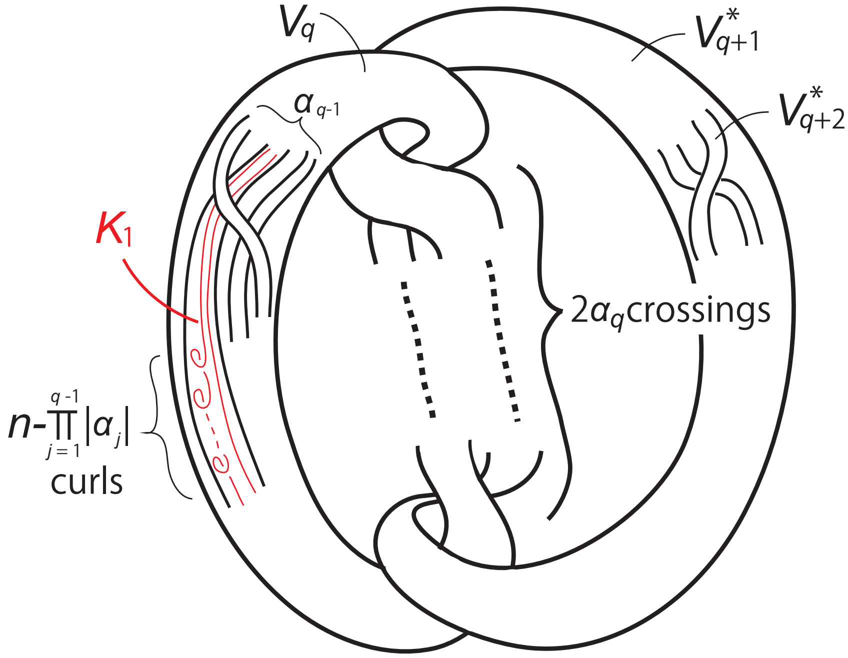


Figure 24:

Proof. We take the highest saddle of T_{q+1}, and denote it by σ. Assume that σ is a non-nested saddle. Let c_1, c_2 be the simple closed curves as in the definition of nested (or non-nested) saddle. Then, we denote the pairwise disjoint meridian disks of V_{q+1} bounded by c_1 and c_2, which are contained in the level sphere S^{ε}_{σ} by D_1 and D_2. Then, by Assertion 4.1, each disk D_i $(i = 1, 2)$ intersects K_1 in at least $|\prod_{j=1}^{q} \alpha_j|$ points. This implies $b(K_1) \geq |\prod_{j=1}^{q} \alpha_j|$, but this contradicts the assumption that $n < |\prod_{j=1}^{q} \alpha_j|$. Hence σ is nested. Then, by Lemmata 4.2 and 4.3, all of the saddles in T_{q+1} are essential and nested. □

Suppose $|\prod_{j=1}^{q-1} \alpha_j| \leq n < |\prod_{j=1}^{q} \alpha_j|$. By Lemma 4.6, and Remark 4.4, we see that V^*_{q+1} looks like a small regular neighborhood of a trivial knot. This together with $k(V^*_{q+1}, K^*_m) = |\prod_{j=q+1}^{m-1} \alpha_j|$ (from Assertion 4.1) shows if T is taut with respect to n-bridge K_1, then $b(K^*_m) \geq |\prod_{j=q+1}^{m-1} \alpha_j|$. This implies that $b_{K_1=n}([L]) \geq n + |\prod_{j=q+1}^{m-1} \alpha_j|$. This together with Lemma 4.5 shows that $b_{K_1=n}([L]) = n + |\prod_{j=q+1}^{m-1} \alpha_j|$. Hence, we have proven the second conclusion of Theorem 1.4.

The next lemma is immediate from Figure 25.

Lemma 4.7. For each n with $n \geq 1$, we have the following; there exists a position of $L = K_1 \cup K_m^*$ such that $b(K_1) = n$ and $b(K_m^*) = |\prod_{j=2}^{m-1} \alpha_j|$. In particular, we have $b_{K_1=n}([L]) \leq n + |\prod_{j=2}^{m-1} \alpha_j|$.

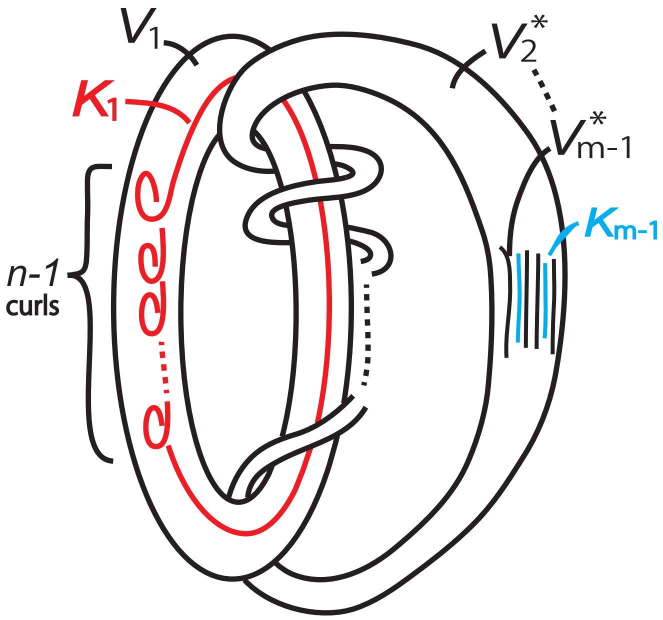


Figure 25:

Lemma 4.8. For each n with $1 \leq n < |\alpha_1|$, we have; if T is taut with respect to n-bridge K_1, then, each saddle of T_2 is nested.

We can prove Lemma 4.8 by using the arguments as in the proof of Lemma 4.6, and we omit describing it. For $1 \leq n < |\alpha_1|$, by Lemma 4.8, Remark 4.4 and Assertion 4.1, we have that if T is taut with respect to n-bridge K_1, then $b(K_m^*) \geq |\prod_{j=2}^{m-1} \alpha_j|$. This implies that $b_{K_1=n}([L]) \geq n + |\prod_{j=2}^{m-1} \alpha_j|$. This together with Lemma 4.7 shows that $b_{K_1=n}([L]) = n + |\prod_{j=2}^{m-1} \alpha_j|$. Hence, we have proven the first conclusion of Theorem 1.4.

Finally, Figure 26 represents a position satisfying $b(K_m^*) = b([K_m^*]) = 1$. Here we note that K_1 in Figure 26 has $|\prod_{j=1}^{m-2} \alpha_j|$ maxima. It means that N in Proposition 1.3 is less than or equal to $|\prod_{j=1}^{m-2} \alpha_j|$. This together with Proposition 1.3 shows that if $n \geq |\prod_{j=1}^{m-2} \alpha_j|$, then $b_{K_1=n}([L]) = n + 1$. This gives the third conclusion of Theorem 1.4, and completes the proof of Theorem 1.4.

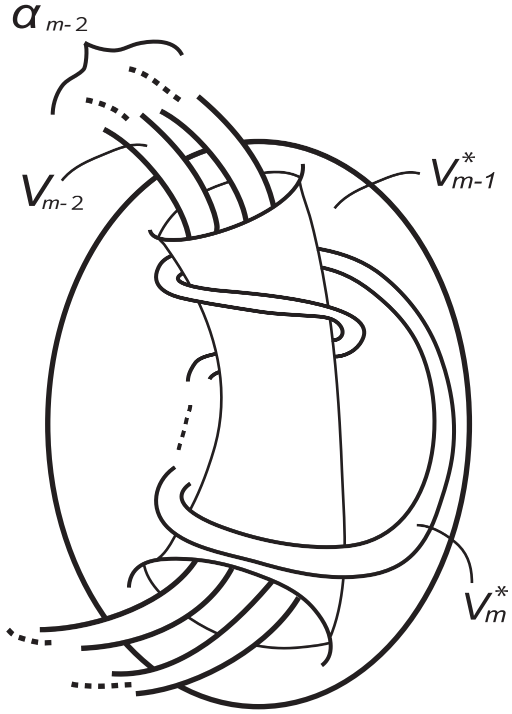


Figure 26:

5 Genus g bridge index and constrained bridge index

In [Z], A. Zupan studies genus g bridge index of links, particularly the sequence of genus g bridge indices $(b_0(K), b_1(K), \dots)$ called bridge spectrum (for the definition of these terms, see below). He presents a kind of interesting behaviors of the bridge spectrum by using iterated torus knot. Note that these are the knots we utilized in Section 4. In this section, we firstly quickly review the result of Zupan's, and by using Heegaard splitting of 3-manifold we propose a viewpoint which unifies the result and constrained bridge indices studied in Section 4.

The union of mutually disjoint arcs $\Gamma = \gamma_1 \cup \cdots \cup \gamma_n$ properly embedded in a 3-manifold M is *trivial* if there is an embedded collection $D_1 \cup \cdots \cup D_n$ of disks in M such that, for each $1 \leq i \leq n$, $\partial D_i \cap \Gamma = \gamma_i$ and $\partial D_i \cap \partial M$ is the arc $\alpha_i = \partial D_i \setminus \mathrm{int}(\gamma_i)$. The collection of arcs $\{\alpha_i\}$ is called *projection* of Γ onto ∂M and the collection of disks is called the *trace disks* of the projection.

A connected 3-manifold C is a *compression body* if there exists a (possibly empty) compact surface F such that C is obtained from $F \times [0,1]$ and a 3-ball B by attaching 1-handles to $(F \times \{1\}) \cup \partial B$. The union of the subsurfaces of ∂C corresponding to $F \times \{0\}$ is denoted by $\partial_- C$. Then $\partial_+ C$ denotes $\mathrm{cl}(\partial C \setminus N(\partial_- C))$. For example, see Figure 27.

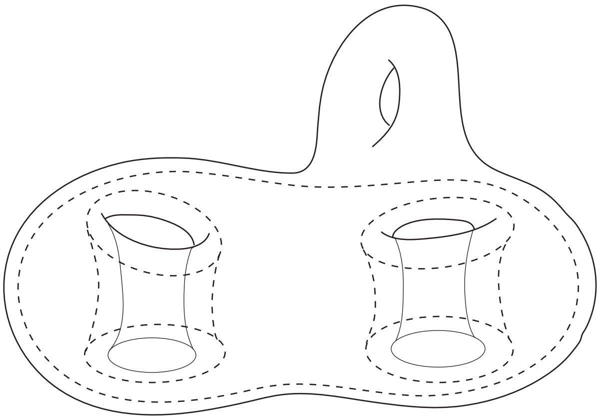


Figure 27: Compression body

Let B_1, B_2 be pairwise disjoint subsurfaces of ∂M such that each component of $\mathrm{cl}(\partial M \setminus (B_1 \cup B_2))$ is an annulus intersecting both B_1 and B_2. Then a surface Σ properly embedded in M is called a *Heegaard surface* of $(M; B_1, B_2)$ if Σ decomposes M into two compression bodies C_1, C_2 such that

$\partial_+ C_i = \Sigma$, and $\partial_- C_i = B_i$ $(i = 1, 2)$. The decomposition $C_1 \cup_\Sigma C_2$ is called a *Heegaard splitting* of $(M; B_1, B_2)$.

Remark 5.1. It is known that each compact orientable 3-manifold with specified subsurfaces of the boundary as above admits a Heegaard surface ([Mo], [CG]).

Note that if M is a closed 3-manifold, each Heegaard splitting of M $(= (M; \emptyset, \emptyset))$ is a decomposition of M into two handlebodies. See Figure 28.

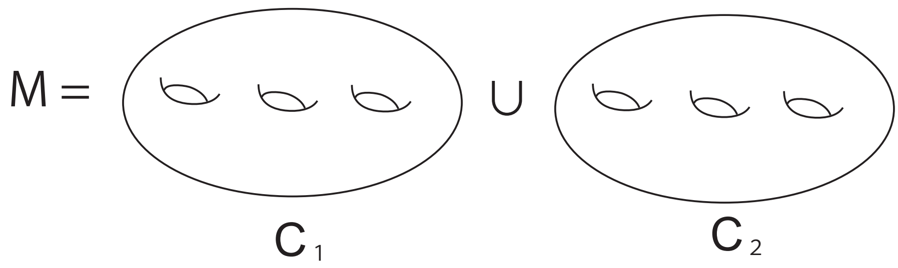


Figure 28: Heegaard splitting of M $(= (M; \emptyset, \emptyset))$

Let L be a link in a closed orientable 3-manifold M with a Heegaard splitting $M = C_1 \cup_\Sigma C_2$. We say that L is in an *n-bridge position* with respect to the Heegaard surface Σ if $L \cap C_i$ $(i = 1, 2)$ is a union of arcs which is trivial in C_i $(i = 1, 2)$. Particularly if Σ is a genus g surface, then we say that L is in a *genus g, n-bridge position*. The *genus g bridge index* of L, denoted by $b_g(L)$, is the smallest integer n, for which L is in an n-bridge position with respect to some genus g Heegaard surface of M. In [Z], Zupan proposed, for a knot K in S^3, to study the sequence $\mathbf{b}(K) = (b_0(K), b_1(K), \dots)$, called *bridge spectrum*, and showed the next theorem which seems to be relevant to Theorem 1.4:

Theorem 5.2. Let K_n be the iterated torus knot associated to $((p_0, q_0), \dots, (p_n, q_n))$. Suppose for each i $(1 \leq i \leq n)$, $|p_i - p_{i-1} \cdot q_{i-1} \cdot q_i| > 1$. Then;

1. for $g < n$, $\quad b_g(K_n) = |q_n| \cdot b_g(K_{n-1})$;
2. for $g = n$, $\quad b_g(K_n) = \min\{|p_n - p_{n-1} \cdot q_{n-1} \cdot q_n|, |q_n|\}$;
3. $\quad b_g(K_n) = 0 \quad$ otherwise.

Example 5.3. **([Z])** Take the iterated torus knot K_1 associated to $((3,2),(21,4))$. Then we have the following;

$$b_0(K_1) = 4 \cdot 2 = 8,$$

$$b_1(K_1) = \min\{|21 - 3 \cdot 2 \cdot 4|, 4\} = 3.$$

Thus we have;

$$\mathbf{b}(K_1) = (8, 3, 0, \dots).$$

Genus g constrained bridge index from the viewpoint of Heegaard splitting

Let $L = K_1 \cup K_2$ be a 2-component link. Since bridge sphere for any link is a genus 0 Heegaard splitting, the sequence of constrained bridge indices and bridge spectrum can be unified as in the following form:

$$b_{K_1=n,g}([L]) := \min\left\{ m \;\middle|\; \begin{array}{l} \text{there exists a genus } g \text{ Heegaard surface } \Sigma \\ \text{such that } L' \text{ is in an } m\text{-bridge position} \\ \text{with respect to } \Sigma, \text{ where } K_1' \text{ is in an} \\ n\text{-bridge position with respect to } \Sigma \end{array} \right\},$$

and we call it *a constrained bridge index with respect to genus g Heegaaard surface*. In the remainder of this paper, we give an alternative presentation of $b_{K_1=n,g}([L])$ using Heegaard splitting.

Let K be a knot in a closed 3-manifold M. Let $T = \partial N(K)$, and $E(K) = M \setminus \text{int}N(K)$. For each $m \geq 1$, let $A_1, B_1, A_2, B_2, \dots, A_m, B_m$ be mutually disjoint meridional annuli in T, which are arrayed in T in this order. See Figure 29. Further let $\mathcal{A}_m = A_1 \cup \cdots \cup A_m$, and $\mathcal{B}_m = B_1 \cup \cdots \cup B_m$. Then, from the definition of Heegaard splitting of $(E(K); \mathcal{A}_m, \mathcal{B}_m)$, we immediately have the following. (Figure 30 is a key observation of the proof.)

Proposition 5.4. The knot K admits a genus g, n-bridge position if and only if $(E(K); \mathcal{A}_n, \mathcal{B}_n)$ admits a genus g Heegaard surface.

Hence the genus g bridge index of K, denoted by $b_g(K)$ above, is expressed as in the following form;

$$b_g(K) = \min\{n \mid (E(K); \mathcal{A}_n, \mathcal{B}_n) \text{ admits a genus } g \text{ Heegaard surface}\}.$$

Let $L = K_1 \cup K_2$ be a link, and $E = S^3 \setminus \text{int}N(K_1)$. Let $T = \partial E$, and let $\mathcal{A}_n, \mathcal{B}_n$ $(\subset T)$ be as above. Then let

$$b_{(E;\mathcal{A}_n,\mathcal{B}_n)}(K_2) = \min\left\{ \ell \;\middle|\; \begin{array}{l} K_2 \text{ admits } \ell \text{ bridge position with respect to} \\ \text{genus 0 Heegaard surface of } (E; \mathcal{A}_n, \mathcal{B}_n) \end{array} \right\}.$$

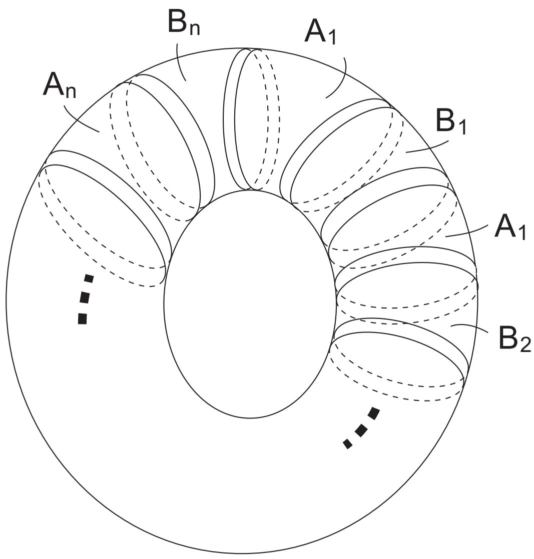
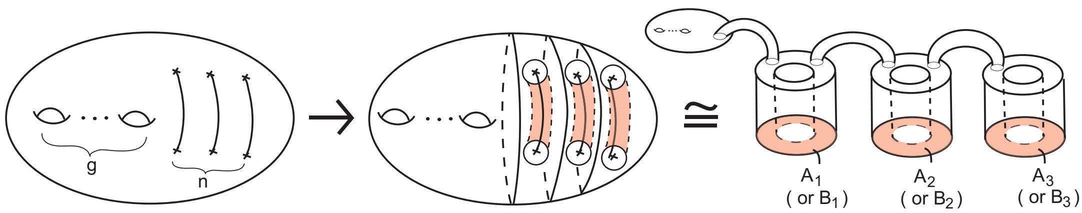
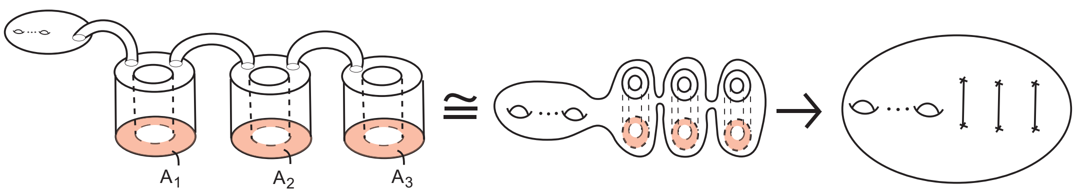


Figure 29:



an idea of a proof of Proposition 5.4



Heegaard splitting of (E ; $\mathcal{A}_n$, $\mathcal{B}_n$) induces a bridge position

Figure 30: $E(K)$ is separated into compression bodies

Then the constrained bridge index of L is expressed as in the following form;

$$b_{K_1=n}(L) = n + b_{(E;\mathcal{A}_n,\mathcal{B}_n)}(K_2).$$

We introduce the following notations. Let $E' = E \setminus \mathrm{int}N(K_2)$. For each $m \geq 1$, let $C_1, D_1, C_2, D_2, \ldots, C_m, D_m$ be mutually disjoint meridional annuli in $\partial N(K_2)$, which are arrayed in $\partial N(K_2)$ in this order. Then from the above arguments, we see that $b_{K_1=n,g}([L])$ can be expressed as in the following form.

$$b_{K_1=n,g}([L]) = \min \left\{ m \;\middle|\; \begin{array}{c} (E'; \mathcal{A}_n \cup \mathcal{C}_m, \mathcal{B}_n \cup \mathcal{D}_m) \text{ admits} \\ \text{a genus } g \text{ Heegaard surface} \end{array} \right\}.$$

Hence it is interesting to study the Heegaard genus of $(E'; \mathcal{A}_n \cup \mathcal{C}_n, \mathcal{B}_n \cup \mathcal{D}_n)$ for $(n, m) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$.

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