Pointwise multipliers on some function spaces

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CONTENTS

Introduction

Let 9 ∈ Loo(Rn). For any f ∈ LP(An), the pointwise multiplication fg is in LP(R^n) by a property of the Lebesgue integral. Conversely, it is well known that if fg is in LP(Rn) for any f ∈ LP(Rn) then 9 ∈ Loo(Rn).

The space of functions of bounded mean oscillation, BMO, is introduced by John and Nirenberg [11](1961). BMO(Rn) includes Loo(Rn) and is included in $L^p_{\mathrm{loc}}(\mathbb{R}^n)$. The theory of this space has been developed by many authors, Carleson, Coifman, Fefferman, Janson, Jones, Reimann, Spanne, Stein, Uchiyama, etc. Unfortunately, for f ∈ BMO(Rn) and for 9 ∈ Loo(Rn), fg is not necessarily in BMO(An). So, it seems to be meaningful to investigate the pointwise multiplications on BMO(Rn).

Let A be a function space and 9 be a function. If f E A ⇒ f9 E A, then 9 is called a pointwise multiplier on A. We denote the set of all pointwise multipliers on A by PWM(A). With this notation we can state the example at the beginning as follows:

$$PWM(L^p(\mathbb{R}^n)) = L^\infty(\mathbb{R}^n).$$

Johnson [12](1975) showed that

$$PWM(BMO(R)) \subsetneq LOO(R).$$

The pointwise multipliers on BMO are not so simple as on LP.

On the torus T, local structures of function spaces are reflected on the pointwise multipliers. Let $\Lambda_\alpha(\mathbb{T})$, $0 < \alpha \leq 1$, be the space of a-Lipschitz continuous functions, and let Ck(T) be the space of k-times continuously differentiable functions. Then

$$PWM(\Lambda_\alpha(\mathbb{T})) = \Lambda_\alpha(\mathbb{T}),$$

and

$$PWM(C^k(\mathbb{T})) = C^k(\mathbb{T}).$$

Stegenga [29](1976) has characterized the pointwise multipliers on BlvIO(T). Using this characterization, he could characterize a class of bounded Toeplitz operators on HI(T) by use of the fact that the dual space of HI is BMO.

This Hardy space H^1, which is a subspase of L^1, is also an important function space and has been studied by many authors. Using the duality, he showed that

$$PWM(H^1(\mathrm{T})) = PWM(BMO(\mathrm{T})).$$

However, in contrast with this,

$$PWM(H^1(\mathrm{R})) = \{\text{constant functions}\}.$$

Janson [10](1976) has characterized the pointwise multipliers on $BMO_\phi(\mathrm{T}^n)$, where ϕ is assumed to be a positive non-decreasing concave function defined on $(0,1)$. If $\phi(r) = r^\alpha$ then $BMO_\phi(\mathrm{T}^n) = \Lambda_\alpha(\mathrm{T}^n)$, and if $\phi(r) = 1$ then $BMO_\phi(\mathrm{T}^n) = BMO(\mathrm{T}^n)$. Therefore Janson's characterization is a generalization of Stegenga's one. It is the following:

$$PWM(BMO_\phi(\mathrm{T}^n)) = L^\infty(\mathrm{T}^n) \cap BMO_\psi(\mathrm{T}^n),$$

where

$$\psi(r) = \phi(r) \bigg/ \int_r^1 \frac{\phi(t)}{t}\, dt\ .$$

This result shows that the pointwise multipliers reflect deeply the local structure of $BMO_\phi(\mathrm{T}^n)$.

It is interesting that there are two cases, the first one is that the pointwise multipliers reflect deeply the structure of the function space, and the second is that they don't reflect so deeply.

Our main theme is to study the pointwise multipliers on some function spaces, defined using the mean oscillation, whose structures are reflected deeply on the pointwise multipliers.

Let (X, μ) be a measure spase, and let $Meas(X)$ be the set of all measurable functions defined on X. We consider a function as an element modulo null-functions. Then, for $f, g \in Meas(X)$, the pointwise multiplication fg is well defined and

$$PWM(Meas(X)) = Meas(X).$$

Usually, BMO is considerd as a space modulo constants. But the pointwise multipliers are defined on the function spaces or on the spaces modulo null-functions. To consider pointwise multipliers, we denote bmo instead of BMO and treat as a space modulo null-functions.

In chapter I, we investigate the structure of the function spaces on which pointwise multipliers are bounded functions, and we compare with the structure of the space of functions of bounded mean oscillation. We consider some Banach spaces and complete quasi-normed linear spaces of functions defined on a measure space (X, μ), and we state sufficient conditions that the set of all pointwise multipliers equals $L^\infty(X)$.

BMO_ϕ, its generalization $L^{p,\Phi}$, measure weighted BMO, Morrey spaces and Lipschitz spaces have been studied by many authers. In chapter II, to generalize these spaces, we introduce a function space $bmo_{w,p}(\mathbb{R}^n)$. It is defined using the mean oscillation in L^p-sense ($1 \le p < \infty$) and a weight function $w : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$. We characterize the pointwise multipliers on this function space. The pointwise multipliers reflect deeply the structure of this function space.

In chapter III, we deepen our study by treating the pointwise multipliers on $bmo_\phi(\mathbb{R}^n)$ which is a special case of $bmo_{w,p}(\mathbb{R}^n)$, where ϕ depends only on $r \in \mathbb{R}_+$. Our characterization shows that the pointwise multipliers on $bmo_\phi(\mathbb{R}^n)$ reflect deeply not only the local structure but also the global structure of this space. Therefore we need a weight functon w depending on $a \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$ introduced in the previous chapter. Moreover, we state some sufficient conditions for the pointwise multipliers on $bmo_\phi(\mathbb{R}^n)$, and we give examples of the pointwise multipliers on this space. Next we consider the pointwise multipliers on subspaces of $bmo_\phi(\mathbb{R}^n)$ by contrast with $bmo_\phi(\mathbb{T}^n)$, on which the pointwise multipliers reflect the only local structure. At the end of this chapter, we characterize the pointwise multipliers on the local Hardy space $h^1(\mathbb{R}^n)$ introduced by Goldberg [8](1979), whose dual space is a subspace of $bmo_\phi(\mathbb{R}^n)$.

In chapter IV, we characterize the pointwise multipliers on Morrey spaces and on the spaces of functions of bounded mean oscillation with the Muckenhoupt A_p-weight. One of the latter is a generalized Morrey space.

Let A and B be linear spaces of functions defined on a measure space. In chapter V, we consider the set of all pointwise multipliers from A to B, $PWM(A, B)$. It is well known that, for $1/p = 1/p_1 + 1/p_2$, $1 \le p \le \infty$,

$$PWM(L^{p_1}(\mathbb{R}^n), L^p(\mathbb{R}^n)) = L^{p_2}(\mathbb{R}^n).$$

We generalized this equality to Lorentz spaces as follows:

$$PWM(L^{(p_1,q_1)}(X), L^{(p,q)}(X)) = L^{(p_2,q_2)}(X).$$

I. Pointwise multipliers on some Banach spaces and complete quasi-normed linear spaces

It is well known that, for $1 \leq p \leq \infty$,

$$PWM(L^p(\mathbb{R}^n)) = L^\infty(\mathbb{R}^n).$$

In this chapter, to compare with the space of functions of bounded mean oscillation, we investigate structures of function spaces on which pointwise multipliers are bounded functions. And we generalize the above equality to Banach spaces and complete quasi-normed linear spaces.

1. Quasi-normed linear spaces.

Let (X, μ) be a measure spase and let A be a linear space of functions defined on X. The following is a necessary and sufficient condition for $L^\infty(X) \subset PWM(A)$.

(1.1.1) $\qquad f \in A \quad \text{and} \quad |h(x)| \leq |f(x)| \quad \text{a.e. } X \qquad \Rightarrow \qquad h \in A.$

If A has a norm $\|\cdot\|$ and satisfies

(1.1.2) $\quad f \in A \quad \text{and} \quad |h(x)| \leq |f(x)| \quad \text{a.e. } X$

$$\Rightarrow \qquad h \in A \quad \text{and} \quad \|h\| \leq \|f\|,$$

then any function $g \in L^\infty(X) \subset PWM(A)$ is a bounded operator and

$$\|g\|_{op} \leq \|g\|_{L^\infty},$$

where $\|g\|_{op}$ is the operator norm of the pointwise multiplier g.

On the other hand, if X is σ-finite, and if A is a Banach space \vhich satisfies Chebyshev's inequality, i.e.

(1.1.3) $\qquad\qquad | \qquad | \qquad\qquad \dfrac{|\;\;|}{\eta} \quad \text{for } \eta > 0,\ f \in A,$

then any pointwise multiplier on A is a bounded operator.

We concider the pointwise multipliers on complete quasi-normed linear spaces, which are generalizations of Banach spaces.

Assume that $A \subset Meas(X)$ and A is a quasi-normed linear space, i.e. there is a constant $\kappa \geq 1$ such that:

In this case, there is a distance $d(f,g) = d(f-g)$ depending only on $f-g$ such that

$$d(f,g) \leq \|f-g\|^p \leq 2d(f,g),$$

where $0 < p \leq 1$, $\kappa = 2^{(1/p)-1}$. A linear operator T defined on A into A is continuous with respect to the distance d, if and only if there is a constant $\beta > 0$ such that

$$\|Tf\| \leq \beta \|f\| \quad \text{for all } f \in A.$$

Then T is called bounded. We define

$$\|T\| = \|T\|_{op} = \inf\{\beta : \|Tf\| \leq \beta \|f\| \text{ for all } f \in A\}.$$

For a complete quasi-normed linear space, we have the uniform boundedness theorem and the closed graph theorem.

THEOREM (THE UNIFORM BOUNDEDNESS THEOREM). *Let A be a complete quasi-normed linear space. Let a family $\{T_\lambda; \lambda \in \Lambda\}$ of bounded operators be defined on A into A. If the set $\{T_\lambda f; \lambda \in \Lambda\}$ is bounded at each $f \in A$, then $\{\|T_\lambda\|; \lambda \in \Lambda\}$ is bounded.*

THEOREM (THE CLOSED GRAPH THEOREM). *Let A be a complete quasi-normed linear space. A closed linear operater defined on A into A is bounded.*

At the end of this section, we generalize the notion of Chebyshev's inequality. Let (X,μ) be a σ-finite measure space, i.e. X is expressible as a countable union of sets $X_i \subset X$ such that $\mu(X_i) < \infty$ ($i = 1,2,\dots$). Let $A \subset Meas(X)$ be a complete quasi-normed linear space. And let $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$ ($i = 1,2,\dots$) be nondecreasing functions and $\Phi_i(t) \to 0$ $(t \to 0)$ for each i. Then we define the generalized Chebyshev's inequality as follows:

$$(1.1.4) \quad \mu(\{x \in X_i;\ |f(x)| \ge \eta\}) \le \Phi_i\left(\frac{\|f\|}{\eta}\right) \quad \text{for } \eta > 0,\ f \in A,\ i = 1,2,\ldots.$$

2. Sufficient conditions for $PWM(A) = L^\infty(X)$.

For $1 \le p \le \infty$, $L^p(R^n)$ is a Banach space, and for $0 < p < 1$, $L^p(R^n)$ is a complete quasi-normed linear space. $L^p(R^n)$ satisfies the conditions both (1.1.2) and (1.1.3). To show $PWM(L^p(R^n)) = L^\infty(R^n)$, we can use these conditions. On the other hand, $bmo_{w,p}(R^n)$ defined in the next chapter do not satisfy the condition (1.1.1), i.e. $L^\infty(R^n)$ is not included in $PWM(bmo_{w,p}(R^n))$. But $bmo_{w,p}(R^n)$ satisfies the condition (1.1.4). It follows from Lemmas 1.3 and 1.4 in this section that the pointwise multiplier on $bmo_{w,p}(R^n)$ is a bounded operator. Moreover $PWM(bmo_{w,p}(R^n)) \subset L^\infty(R^n)$.

Let (X, μ) be a measure space and let A be a linear space of functions defined on X. In this section, we concider sufficient conditions for $PWM(A) = L^\infty(X)$.

THEOREM 1.1. *Let (X, μ) be a σ-finite measure space and let X be a union of sets $X_i \subset X$ such that $\mu(X_i) < \infty$ $(i = 1, 2, \ldots)$. Let $A \subset Meas(X)$ be a complete quasi-normed linear space. Assume that any bounded function whose support is included in X_i for some i is in A. If A satisfies the condition (1.1.2), then*

$$PWM(A) = L^\infty(X) \qquad \text{and} \qquad \|g\|_{Op} = \|g\|_{L^\infty}.$$

PROOF OF THEOREM 1.1. Assume that g is in $L^\infty(X)$. For any $f \in A$,

$$|fg(x)| \le \|g\|_{L^\infty} |f(x)| \quad \text{a.e. } X$$

It follows from (1.1.2) that fg is in A and that $\|fg\|_A \le \|g\|_{L^\infty} \|f\|_A$. Therefore g is in $PWM(A)$ and

$$\|g\|_{Op} \le \|g\|_{L^\infty}.$$

Next we show

$$(1.2.1) \qquad \|g\|_{Op} \ge \|g\|_{L^\infty}.$$

We can asuume that $\|g\|_{L^\infty} \neq 0$. For any η, $0 < \eta < \|g\|_{L^\infty}$, we choose X_i such that

$$0 < \mu(E_\eta \cap X_i) < \infty \quad \text{where} \quad E_\eta = \{x \in X : |g(x)| > \eta\}.$$

Let h_η be the characteristic function of $E_\eta \cap X_i$. Then

$$\eta |h_\eta(x)| \leq |h_\eta g(x)| \quad \text{a.e. } X.$$

Since $h_\eta g$ is bounded and its support is included in X_i, $h_\eta g$ is in A. It follows from (1.1.2) that

$$\eta \leq \frac{\|h_\eta g\|_A}{\|h_\eta\|_A} \leq \|g\|_{Op}.$$

Thus we have (1.2.1), and

$$\|g\|_{Op} = \|g\|_{L^\infty}.$$

Conversely, assume that g is in $PWM(A)$. Let

$$g_n(x) = \begin{cases} g(x), & \text{if } |g(x)| \leq n, \\ n, & \text{if } |g(x)| > n, \end{cases}$$

then g_n is in L^∞. By the first half of the proof,

$$\|g_n\|_{L^\infty} = \|g_n\|_{Op}.$$

For any $f \in A$,

$$fg \in A \quad \text{and} \quad |fg_n(x)| \leq |fg(x)| \quad \text{a.e. } X.$$

It follows from (1.1.2) that fg_n is in A and that

$$\|fg_n\|_A \leq \|fg\|_A \quad \text{for any} \quad n.$$

By the uniform boundedness theorem, there is a constant $M, 0 < M < \infty$, such that

$$\sup_n \|g_n\|_{L^\infty} = \sup_n \|g_n\|_{Op} = M.$$

Therefore g is in $L^\infty(X)$.

THEOREM 1.2. *Let (X, μ) be a σ-finite measure space and let $A \subset Meas(X)$ be a complete quasi-normed linear space. Assume that, for almost every $x \in X$, there is a sequence $\{h_{x,n}\}_{n=1}^{\infty}$ of functions in A such that, if $gh_{x,n} \in A$ then*

(1.2.2)

where C is a positive constant independent of $x \in X$. If A satisfies the conditions (1.1.2) *and* (1.1.4), *then*

$$PWM(A) \subset L^{\infty}(X) \quad \text{and} \quad \|g\|_{Op} \le \|g\|_{L^\infty} \le C\|g\|_{Op}.$$

To prove Theorem 1.2, we show Lemmas.

LEMMA 1.3. *Let (X, μ) be a σ-finite measure space and let X be a union of sets $X_i \subset X$ such that $\mu(X_i) < \infty$ $(i = 1,2,\dots)$. Let $A \subset Meas(X)$ be a quasi-normed linear space. If A satisfies the condition* (1.1.4), *then*

$$(1.2.3) \quad f_n \to f \ \text{ in } A \quad \Rightarrow$$
$$\forall i \quad \exists \{f_{n(j)}\}_{j=1}^{\infty}, \text{ a subsequence of } \{f_n\}_{n=1}^{\infty}, \quad \text{s.t. } f_{n(j)} \to f \ \text{ a.e. } X_i$$

PROOF. Assume that $f_n \to f$ in A. For each i, and for all $\eta > 0$,

$$\mu(\{x \in X_i;\ |f_n(x) - f(x)| \ge \eta\}) \le \Phi_i\left(\left\| \frac{|f_n - f|}{\eta} \right\|\right) \to 0 \quad \text{as } n \to \infty,$$

i.e. f_n tend to f with respect to the measure $\mu|X_i$. Then there is a subsequense $f_{n(j)}$ $(j = 1,2,\dots)$ such that

$$f_{n(j)}(x) \to f(x) \quad \text{a.e. } X_i.$$

LEMMA 1.4. *Let (X, μ) be a σ-finite measure space and let X be a union of sets $X_i \subset X$ such that $\mu(X_i) < \infty$ $(i = 1,2,\dots)$. Let $A \subset Meas(X)$ be a complete quasi-normed linear space. If A satisfies the condition* (1.2.3), *then any pointwise multiplier on A is a bounded operator.*

PROOF. Assume that g is in $PWM(A)$. We show g is a closed operator. If

$$f_n \to f \ \text{ in } A \quad \text{and} \quad f_n g \to v \ \text{ in } A,$$

then, it follows from (1.2.3) that

$$1n(j) \to 1 \quad \text{a.e. } Xi \qquad \text{and} \qquad In(j(k))g \to v \quad \text{a.e. } Xi.$$

Therefore $1_9 = v$ a.e. Xi for all i. And we have $1_9 - v$ a.e. X. By the closed graph theorem, 9 is a bounded operator.

PROOF OF THEOREM 1.2. Assume that 9 is in $LOO(X)$. For any $1 \in A$,

$$|1g(x)| \leq 9 \|_{LOO} \, I(x) \quad \text{a.e. } X.$$

It follows from (1.1.2) that Ig is in A and that $Ig_{A} \leq 9 \|_{LOO} II_{A}$. Therefore 9 is in $PWM(A)$ and

$$\|g\|_{Op} \leq \|g\|_{L^\infty}.$$

Conversely, assume that 9 is in $PWM(A)$. It follows from Lemmas 1.3 and 1.4 that 9 is a bounded operator. For $h_{x,n}$ E A, we have

$$\frac{|\;\;|}{|h_{x,n}|_{A}} - |_{gOp}.$$

It follows from (1.2.2) that

$$Ig(x) \leq C| 9_{Op} \quad \text{a.e. } X.$$

Therefore 9 is in $LOO(X)$ and that

$$\|g\|_{L^\infty} \leq C\|g\|_{Op}.$$

II. Pointwise multipliers on $bmo_{w,p}(\mathbb{R}^n)$

The purpose of this chapter is to characterize the pointwise multipliers on $bmo_{w,p}(\mathbb{R}^n)$, which is the function space defined using the mean oscillation in L^p-sense$(1 \leq p < \infty)$ and a weight function $w(x,r) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$. The pointwise multipliers reflect deeply the structure of this space. $bmo_{w,p}(\mathbb{R}^n)$ is a generalization of many function spaces, $bmo_\phi(\mathbb{R}^n)$, $L^{p,\Phi}(\mathbb{R}^n)$, measure weighted BMO-spaces, Morrey spaces and α-Lipschitz spaces, etc. ... .

In the first section we state the definition of $bmo_{w,p}$. In the second section we state the mein theorem. The third and fourth sections are for the preliminaries and lemmas. In the last section we give a proof of the theorem.

The letter C will always denote a constant, not necessarily the same one.

1. Definitions.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. For $a \in \mathbb{R}^n$ and for $r > 0$, let $I(a,r)$ be the cube $\{x \in \mathbb{R}^n : |x_i - a_i| \leq r/2, i = 1,2,\ldots,n\}$ whose edges have length r and are parallel to the coordinate axes. We denote the Lebesgue measure of $E \subset \mathbb{R}^n$ by $|E|$. For a function $f \in L_{loc}(\mathbb{R}^n)$ and for a cube $I = I(a,r)$, we denote the mean value and the mean oscillation of f on I by

$$f_I = M(f,I) = \frac{1}{|I|}\int_I f(x)\,dx$$

and

$$\frac{1}{|I|}\int_I |f(x) - f_I|\,dx$$

respectively. Let $\mathbb{R}_+$ be the set of all positive real numbers. For $1 \leq p < \infty$ and for a weight function $w(x,r) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$, we denote the weighted mean oscillation of $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ on I by

$$MO_{w,p}(f,I) = MO_{w,p}(f,a,r) = \left(\frac{1}{w(I)}\int_I |f(x) - f_I|^p\,dx\right)^{1/p}$$

where $w(I) = w(a,r)$ for a cube $I = I(a,r)$.

Now we define

$$bmo_{w,p}(\mathbb{R}^n) = \left\{ f \in L^1_{loc}(\mathbb{R}^n) : \sup_I MO_{w,p}(f, I) < \infty \right\},$$

$$\|f\|_{BMO_{w,p}} = \sup_I MO_{w,p}(f, I), \qquad \|f\|_{bmo_{w,p}} = \|f\|_{BMO_{w,p}} + M(f, 0, 1).$$

If there is a positive constant C such that $w(I) \leq Cv(I)$ for any I, then $bmo_{w,p}(\mathbb{R}^n) \subset bmo_{v,p}(\mathbb{R}^n)$. Therefore if w and v are comparable i.e. there is a positive constant C such that

$$C^{-1} \leq w(I)/v(I) \leq C \quad \text{for any } I,$$

then $bmo_{w,p}(\mathbb{R}^n) = bmo_{v,p}(\mathbb{R}^n)$.

Usually, $bmo_{w,p}$ denoded by $BMO_{w,p}$ equipped with the seminorm $\|\cdot\|_{BMO_{w,p}}$. Then $BMO_{w,p}$ modulo constants is a Banach space. But the pointwise multipliers are defined on the function spaces or on the spaces modulo null-functions. To consider pointwise multipliers, we denote $bmo_{w,p}$ instead of $BMO_{w,p}$ and treat as a space modulo null-functions. $bmo_{w,p}$ is a Banach space equipped with the norm $\|\cdot\|_{bmo_{w,p}}$.

$bmo_{w,p}(\mathbb{R}^n)$ is a generalization of many function spaces as follows:

(a) Let $p = 1$ and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function. For $w(x, r) = r^n \phi(r)$,

$$bmo_{w,p}(\mathbb{R}^n) = bmo_\phi(\mathbb{R}^n),$$

(see Campanato [3,4,5] and Spanne [26])

(b) Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function. For $w(x, r) = \Phi(r)$,

$$bmo_{w,p}(\mathbb{R}^n) = L^{p,\Phi}(\mathbb{R}^n),$$

(see Peetre [25]). In particular, let $w(x, r) = r^\lambda$.

(b1) If $\lambda = 0$ then

$$bmo_{w,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n) + \{\text{constants}\}.$$

(b2) If $0 < \lambda < n$ then

$$bmo_{w,p}(\mathbb{R}^n) = \mathcal{M}_{p,\lambda}(\mathbb{R}^n).$$

This is the Morrey space (see Morrey [16], Zorko [31] etc.).

(b3) If $\lambda = n$ then

$$bmo_{w,p}(\mathbb{R}^n) = bmo(\mathbb{R}^n).$$

This is the space of functions of bounded mean oscillation, BMO, introduced by John and Nirenberg [11]. If $f \in bmo(\mathbb{R}^n)$ then, for any cube $I = I(a,r)$,

$$|\{x \in I : |f(x) - f_I| \geq \sigma\}| \leq B \exp(-b\sigma/\|f\|_{BMO})\, r^n \quad \text{for } \sigma > 0,$$

where B, b are constants depending only on n. This John and Nirenberg's inequality and Holder's inequality show that

$$bmo_{w,p}(\mathbb{R}^n) = bmo_{w,1}(\mathbb{R}^n) \quad \text{for } 1 \leq p < \infty, w(x,r) = r^n.$$

(b4) If $n < \lambda \leq n + p$ then

$$bmo_{w,p}(\mathbb{R}^n) = \Lambda_\alpha(\mathbb{R}^n),$$

where $\alpha = (\lambda - n)/p$. This is the space of a-Lipschitz continuous functions (see Meyers [15]).

(c) Let $p = 1$ and u be a doubling measure on $\mathbb{R}^n$, i.e. u be a non-negative locally integrable function satisfying the property:

$$\int_J u(x)\,dx \leq C \int_I u(x)\,dx \quad \text{whenever } I \subset J \text{ and } |J| \leq 2|I|,$$

where C is independent of I and J. For $w(I) = \int_I u(x)\,dx$,

$$bmo_{w,p}(\mathbb{R}^n) = bmo_u(\mathbb{R}^n).$$

This is the measure weighted BMO space.

2. Main theorem.

For $f \in bmo_{w,p}(\mathbb{R}^n)$, it follows from Holder's inequality that

$$MO(f,O,r) = \frac{1}{r^n} \int_{I(O,r)} |f(x) - M(f,O,r)|\,dx$$

$$\leq \left(\frac{1}{r^n} \int_{I(O,r)} |f(x) - M(f,O,r)|^p\,dx \right)^{1/p}$$

$$\leq \left(\frac{w(O,r)}{r^n} \right)^{1/p} \|f\|_{bmo_{w,p}}.$$

Then, for $\eta > 0$, we have

$$\eta\,|\{x \in I(0,r) : |f(x)| \geq \eta\}| \leq \int_{I(0,r)} |f(x)|\,dx$$

$$\leq \left|\int_{I(0,r)} |f(x) - M(f,0,r)|\,dx + r^n |M(f,0,r)|\right|$$

$$\leq r^n \left| MO(f,0,r) + M(f,0,r) - \int_{I(0,1)} f(x)dx + \left|\int_{I(0,1)} f(x)\,dx\right|\right|$$

$$\leq r^n \left| MO(f,0,r) + \left|\int_{I(0,r)} |f(x) - M(f,0,r)|\,dx + M(f,0,1)\right|\right|$$

$$\leq r^n\left((1 + r^n)MO(f,0,r) + |M(f,0,1)|\right)$$

$$\left|\qquad \left|\frac{w\ \circ\ r}{r}\right|^{1/p} \qquad\right| \, \|f\|_{bmo_{w,p}}.$$

Therefore we have the generalized Chebyshev's inequality (1.1.4) in Chapter **I.** It follows from Lemmas 1.3 and 1.4 in Chapter I that any pointwise multiplier on $bmo_{w,p}(\mathbb{R}^n)$ is a bounded operator.

Our main result in this chapter is the following.

THEOREM 2.1. *Let* $1 \leq p < \infty$. *Assume that there exists a constant* $A > 0$ *such that for any* $a, b \in \mathbb{R}^n$, $r > 0$, $s \geq 1$,

(2.2.1) $$A^{-1} \leq w(a,r)/w(a,2r) \leq A,$$

(2.2.2) $$\left|\ \left|^{r} \frac{t\ \ 1/p}{\qquad}\right|^{p}\right.$$

(2.2.3) $$|a - b| \leq r \Rightarrow A^{-1} \leq w(a,r)/w(b,r) \leq A,$$

(2.2.4) $$w(a, sr) \leq A s^{n+p} w(a,r).$$

Then

$$PWM(bmo_{w,p}(\mathbb{R}^n)) = bmo_{w^*,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$$

where $w^* = w/\Psi$, $\Psi = \Psi_1 + \Psi_2$ *and*

(2.2.5) $$\Psi_1(a,r) = \left|\ \int_1 \frac{\qquad}{t^{n/p+1}}\ \right|,$$

(2.2.6) $$\Psi_2(a,r) = \left|\int_r^{\max(2,|a|,r)} \frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt\right|^p.$$

Moreover, the operator norm of $g \in PWM(bmo_{w,p}(R^n))$ *is comparable to*

$$\|g\|_{BMO_{w_*,p}} + \|g\|_\infty.$$

This result shows that the pointwise multipliers reflect deeply the structure of *bmow,p(Rn)*.

Janson [10](1976) has characterized pointwise multipliers on $bmo_\phi(\mathbf{T}^n)$ on the n-dimensional torus Tn, where ϕ is nondecreasing and there is a constant $A > 0$ such that $\phi(r_2)/r_2 \le A\phi(r_1)/r_1$ for $0 < r_1 \le r_2$. In this case, Ψ in our theorem is the following:

$$\Psi(r) = \int_r^1 \frac{\phi(t)}{t}\, dt.$$

In the next chapter, we will consider the case of $bmo_\phi(\mathbf{R}^n)$, which is a special case of *bmow,p(Rn)*. ϕ depends only on $r \in \mathbf{R}_+$. However, Ψ depends not only on $r \in \mathbf{R}_+$ but also on $a \in \mathbf{R}^n$. The pointwise multipliers on $bmo_\phi(\mathbf{R}^n)$ reflect not only the local structure but also the global structure. Therefore we need the weight function depended on $r \in \mathbf{R}_+$ and on $a \in \mathbf{R}^n$.

3. Preliminaries.

In this section, we state some simple lemmas.

LEMMA 2.2.

$$\left(\frac{1}{|I|}\int_I |f(x) - f_I|^p dx \right)^{1/p} \le 2\inf_c \left(\frac{1}{|I|}\int_I |f(x) - c|^p dx \right)^{1/p}.$$

LEMMA 2.3. *If* $|F(z_1) - F(z_2)| \le C|z_1 - z_2|$, *then*

$$MO_{w,p}(F(f), I) \le 2C\, MO_{w,p}(f, I).$$

The above Lemmas are showed by the triangle inequality and Holder's inequality.

LEMMA 2.4. *If* $I_1 \subset I_2$, *then*

(2.3.1) $$|M(f, I_1) - M(f, I_2)| \le \frac{|I_2|}{|I_1|} \ldots$$

and

(2.3.2) $$MO(f, I_1) \leq 2\frac{|I_2|}{|I_1|}MO(f, I_2).$$

PROOF. Since

$$\left| \frac{1}{|I|} \int \qquad \frac{1}{|I|} \int \right|$$

we have (2.3.1). And we have

$$\frac{1}{|I_1|} \int$$

LEMMA 2.5. *There is a constant* $C > 0$ *such that*

(2.3.3) $$|M(f, a, r) - M(f, a, s)| \leq C \int_r^{2s} \frac{MO(f,a,t)}{t}\,dt \quad \text{for } 0 < r < s$$

where C *is independent of* f, a, r *and* s.

PROOF. By (2.3.2), we have

(2.3.4) $$MO(f, a, r) \leq (\log 2)^{-1} \int_r^{2r} \frac{MO(f, a, r)}{t}\,dt \leq C \int_r^{2r} \frac{MO(f, a, t)}{t}\,dt.$$

If $2^{-k-1}s \leq r < 2^{-k}s$, then

$$|M(f, a, r) - M(f, a, s)|$$
$$\leq |M(f, a, r) - M(f, a, 2^{-k}s)| + \sum_{j=0}^{k-1} |M(f, a, 2^{-j-1}s) - M(f, a, 2^{-j}s)|$$
$$\leq 2^n \sum_{j=0}^{k} MO(f, a, 2^{-j}s) \leq C \sum_{j=0}^{k} \int_{2^{-j}s}^{2^{-j+1}s} \frac{MO(f, a, t)}{t}\,dt$$

by (2.3.1) and (2.3.4). This proves (2.3.3).

LEMMA 2.6. *Let* $1 \leq p < \infty$. *There is a constant* $C > 0$ *such that*

$$\int_{|x-a|<r} \left| \int_{|x-a|}^{r} \frac{w(a, t)^{1/p}}{t^{n/p+1}}\,dt \right|^p dx \leq Cw(a, r)$$

where C *is independent of a and r.*

PROOF. We denote the volume of the unit ball by σ_n. Then we have

$$\int_{|x-a|<r}\left(\int_{|x-a|}^{r}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt\right)^{p}dx=\int_0^r\left(\int_\rho^r\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt\right)^{p}\sigma_n\rho^{n-1}\,d\rho$$

$$\leq\left(\int_0^r\left(\int_0^t\sigma_n\rho^{n-1}\,d\rho\right)^{1/p}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt\right)^{p}=\frac{\sigma_n}{n}\left(\int_0^r\frac{w(a,t)^{1/p}}{t}\,dt\right)^{p}$$

$$\leq Cw(a,r)$$

by Minkowski's inequality and (2.2.2).

LEMMA 2.6. *Let $1\leq p<\infty$. There is a constant* C >0 *such that*

$$\int_r^{2s}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt\leq C\int_r^{s}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt\qquad for\ 0<2r\leq s$$

where C *is independent of a, r and s.*

PROOF. By a change of variable and (2.2.1), we have

$$\int_s^{2s}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt=\int_{s/2}^{s}\frac{w(a,2t)^{1/p}}{(2t)^{n/p+1}}\,2\,dt$$

$$\leq\left(\frac{A}{2^n}\right)^{1/p}\int_{s/2}^{s}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt\leq\left(\frac{A}{2^n}\right)^{1/p}\int_r^{s}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt.$$

Therefore

$$\int_r^{2s}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt\leq\left(1+\left(\frac{A}{2^n}\right)^{1/p}\right)\int_r^{s}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt.$$

4. Lemmas.

In this section we show some lemmas needed to prove the theorem. Let $1\leq p<\infty$. First, for $a\in R^n$ and $r>0$, we define

(2.4.1) $$W(a,r)=\int_r^1\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt.$$

LEMMA 2.8. *For $a\in R^n$, let*

$$f_a(x)=W(a,|x-a|).$$

Then $\|f_a\|_{BMO_{w,p}} \le C$ *independently of* a.

PROOF. We show

(2.4.2) $$MO_{w,p}(f_a, b, r) \le C \quad \text{independently of } a, b, \text{ and } r.$$

Case 1: $|a-b| < \sqrt{n}r$. Since $I(b,r) \subset \{|x-a| \le 2\sqrt{n}r\}$, we have

$$\int_{I(b,r)} |f_a(x) - W(a, 2\sqrt{n}r)|^p dx \le \int_{|x-a|\le 2\sqrt{n}r} \left| \int_{|x-a|}^{2\sqrt{n}r} \frac{w(a,t)^{1/p}}{t^{n/p+1}} dt \right|^p dx$$
$$\le Cw(a, 2\sqrt{n}r) \le Cw(b, 2\sqrt{n}r) \le Cw(b,r),$$

by Lemma 2.6, (2.2.1) and (2.2.3). This inequality and Lemma 2.2 show (2.4.2).

Case 2: $|a-b| \ge \sqrt{n}r$. It follows from (2.2.3) and (2.2.4) that

(2.4.3) $$w(a, |a-b|) \le Aw(b, |a-b|) \le A^2 \left| \frac{|a-b|}{r} \right|^{n+p} w(b,r).$$

If $x \in I(b,r)$, then $|x-a|$ is comparable to $|a-b|$. Therefore, for $|x-a| \le t \le |a-b|$ or for $|x-a| \ge t \ge |a-b|$,

(2.4.4) $$\frac{w(a,t)}{t^{n+p}} \le \frac{Cw(a, |a-b|)}{|a-b|^{n+p}}.$$

By (2.4.3) and (2.4.4), we have

$$\int_{I(b,r)} |f_a(x) - W(a, |a-b|)|^p dx = \int_{I(b,r)} \left| \int_{|x-a|}^{|a-b|} \frac{w(a,t)^{1/p}}{t^{n/p+1}} dt \right|^p dx$$
$$\le C\frac{w(b,r)}{r^{n+p}} \int_{I(b,r)} \big| |a-b| - |x-a| \big|^p dx$$
$$\le C\frac{w(b,r)}{r^{n+p}} \int_{I(b,r)} |x-b|^p dx \le Cw(b,r).$$

This inequality and Lemma 2.2 show (2.4.2).

LEMMA 2.9. *Suppose* Ψ *is defined by* (2.2.5) *and* (2.2.6). *Then there is a constant* $C > 0$ *such that*

$$|M(f, a, r)| \le C\|f\|_{bmo_{w,p}} \Psi(a,r)^{1/p}$$

where C *is independent of* $f \in bmo_{w,p}(R^n)$, a *and* r.

PROOF. We show

(2.4.5) $$|M(f, a, r) - M(f, 0, 1)| \le C\|f\|_{BMO_{w,p}} \Psi(a,r)^{1/p}$$

by using (2.3.1), (2.3.3), (2.3.4) and Lemma 2.7.

Case 1: $\max(r,1) \le |a|/2$. Since $I(a,r) \subset I(a,|a|/2) \subset I(0,3|a|)$ and $I(0,1) \subset I(0,3|a|)$, we have

$$|M(f,a,r) - M(f,a,|a|/2)| \le C_1 \int_r^{|a|/2} \frac{MO(f,a,t)}{t}\,dt \le C_1 \|f\|_{BMO_{w,p}} \int_r^{|a|} \frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt$$

and

$$|M(f,a,|a|/2) - M(f,0,3|a|)| + |M(f,0,3|a|) - M(f,0,1)|$$
$$\le C_2\, MO(f,0,3|a|) + C_3 \int_1^{6|a|} \frac{MO(f,0,t)}{t}\,dt \le C_3 \|f\|_{BMO_{w,p}} \int_1^{6|a|} \frac{w(0,t)^{1/p}}{t^{n/p+1}}\,dt$$
$$\le C_4 \|f\|_{BMO_{w,p}} \int_1^{|a|} \frac{w(0,t)^{1/p}}{t^{n/p+1}}\,dt.$$

Hence (2.4.5) follows.

Case 2: $\max(|a|/2, 1) \le r$. Since $I(a,r), I(0,1) \subset I(0,5r)$, we have

$$|M(f,a,r) - M(f,0,1)|$$
$$\le |M(f,a,r) - M(f,0,5r)| + |M(f,0,5r) - M(f,0,1)|$$
$$\le 5^n MO(f,0,5r) + C_5 \int_1^{10r} \frac{MO(f,0,t)}{t}\,dt$$
$$\le C_6 \int_1^{10r} \frac{MO(f,0,t)}{t}\,dt \le C_6 \|f\|_{BMO_{w,p}} \int_1^{10r} \frac{w(0,t)^{1/p}}{t^{n/p+1}}\,dt$$
$$\le C_7 \|f\|_{BMO_{w,p}} \int^{\max(2,r)} \frac{w(0,t)^{1/p}}{t^{n/p+1}}\,dt.$$

Hence (2.4.5) follows.

Case 3: $\max(|a|/2, r) \le 1$. Since $I(a,r), I(0,1) \subset I(a,5)$, we have

$$|M(f,a,r) - M(f,0,1)|$$
$$\le |M(f,a,r) - M(f,a,5)| + |M(f,a,5) - M(f,0,1)|$$
$$\le C_7 \int_r^{5} \frac{MO(f,a,t)}{t}\,dt$$
$$\le C_8 \int_r^{5} \frac{MO(f,a,t)}{t}\,dt \le C_8 \|f\|_{BMO_{w,p}} \int_r^{5} \frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt$$
$$\le C_9 \|f\|_{BMO_{w,p}} \int_r^{2} \frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt.$$

Hence (2.4.5) follows.

The next two lemmas show that the estimate in Lemma 3.2 is sharp.

LEMMA 2.10. *Let*

$$f(x) = \max(-W(0,2), -W(0,|x|)) - \int_1^{\max(2,|x|)} \frac{W(0,t)^{1/p}}{t^{n/p+1}}\,dt.$$

Then $f \in bmo_{w,p}(\mathbb{R}^n)$ *and there is a constant* $C > 0$ *such that*

$$M(f,a,r) \ge C\Psi_1(a,r)^{1/p} \tag{2.4.6}$$

where C *is independent of* $I(a,r)$.

PROOF. It follows from Lemma 2.8 and Lemma 2.3 that $f \in bmo_{w,p}(\mathbb{R}^n)$. Next we show (2.4.6), by using Lemma 2.7 and the fact that $W(0,r)$ is decreasing with respect to r.

Case 1: $4|a| \le r$. Since $\{|x| \le r/4\} \subset I(a,r)$, we have

$$\begin{aligned} M(f,a,r) &\ge r^{-n} \int_{r/8\le|x|\le r/4} f(x)\,dx \\ &\ge r^{-n} \int_{r/8\le|x|\le r/4} \max(-W(0,2), -W(0,r/8))\,dx \\ &\ge C \int_1^{\max(2,r/8)} \frac{W(0,t)^{1/p}}{t^{n/p+1}}\,dt \ge C' \int_1^{8\max(2,r/8)} \frac{W(0,t)^{1/p}}{t^{n/p+1}}\,dt. \end{aligned}$$

This proves (2.4.6).

Case 2: $4|a| \ge r$. Since $I(a,r/(4\sqrt{n})) \subset \{|x| \ge |a|/2\}$, we have

$$\begin{aligned} M(f,a,r) &\ge r^{-n} \int_{I(a,r/(4\sqrt{n}))} f(x)\,dx \\ &\ge r^{-n} \int_{I(a,r/(4\sqrt{n}))} \max(-W(0,2), -W(0,|a|/2))\,dx \\ &= C \int_1^{\max(2,|a|/2)} \frac{W(0,t)^{1/p}}{t^{n/p+1}}\,dt \ge C' \int_1^{8\max(2,|a|/2)} \frac{W(0,t)^{1/p}}{t^{n/p+1}}\,dt. \end{aligned}$$

This proves (2.4.6).

LEMMA 2.11. *For any* $I(a,r)$ *there is* $f \in bmo_{w,p}(\mathbb{R}^n)$ *such that*

$$\|f\|_{bmo_{w,p}} \le C_1 \quad \text{and} \tag{2.4.7}$$

$$M(f,a,r) \ge C_2\Psi_2(a,r)^{1/p} \tag{2.4.8}$$

where $C_1 > 0$ *and* $C_2 > 0$ *are independent of* $1(a,r)$ *and* j.

PROOF. Case 1: $\max(r, 1) \leq |a|/(2\sqrt{n})$. For $1(a,r)$, let

$$j(x) \quad W(a, |x - a|) - M(W(a, |x - a|), 0, 1).$$

Then $M(j, 0, 1) = 0$, so Lemma 2.8 and Lemma 2.3 show (2.4.7). To prove (2.4.8), we note that $W(a,r)$ is decreasing with respect to r. Since $|x - a| \geq |a| - |x| \geq |a| - \sqrt{n}/2 \geq |a|/2$ for $x \in 1(0,1)$, we have

$$M(W(a, |x - a|), 0, 1) \leq W(a, |a|/2).$$

And since $|x - a| \leq \sqrt{n}r/2$ for $x \in 1(a, r)$,

$$M(W(a, |x - a|), a, r) \geq W(a, \sqrt{n}r/2).$$

Therefore, by a change of variable, (2.2.1) and Lemma 2.7, we have

$$M(j, a, r) \geq W(a, \sqrt{n}r/2) - W(a, |a|/2)$$
$$= \int_{\sqrt{n}r/2}^{|a|/2} \frac{w(a, t)^{1/p}}{t^{n/p+1}}\, dt \geq C \int_{r}^{|a|/\sqrt{n}} \frac{w(a, t)^{1/p}}{t^{n/p+1}}\, dt \geq C' \int_{r}^{|a|} \frac{w(a,t)^{1/p}}{t^{n/p+1}}\, dt.$$

This proves (2.4.8).

Case 2: $\max(1, |a|/(2\sqrt{n})) \leq r$. For $1(a, r)$, let

$$j(x) = \max(W(0, 1/(8\sqrt{n})) - W(0, |x|), 0)$$

which is independent of $1(a,r)$. There is a cube $1(b,r/4) \subset 1(a,r) \cap \{|x| \geq r/4\}$. Since $1/(8\sqrt{n}) \leq r/(8\sqrt{n}) \leq r/4 \leq |x|$ for $x \in 1(b,r/4)$, we have

$$j(x) \geq W(0, r/(8\sqrt{n})) - W(0, r/4) = \int_{r/(8\sqrt{n})}^{r/4} \frac{w(0, t)^{1/p}}{t^{n/p+1}}\, dt$$
$$\geq C \int_{r}^{2\sqrt{n}r} \frac{w(0, t)^{1/p}}{t^{n/p+1}}\, dt. \quad \text{for } x \in I(b, r/4)$$

and

$$M(f, a, r) \geq 4^{-n} M(f, b, r/4) \geq 4^{-n} C \int_{r}^{2\sqrt{n}r} \frac{w(0, t)^{1/p}}{t^{n/p+1}}\, dt.$$

For $r \leq t \leq 2\sqrt{n}r$ and for $|a| \leq 2\sqrt{n}r$, $w(0,t)$ is comparable to $w(a,t)$. Then

$$M(j, a, r) \geq C' \int_{r}^{2\sqrt{n}r} \frac{w(a, t)^{1/p}}{t^{n/p+1}}\, dt.$$

This proves (2.4.8).

Case 3: $\max(r, |a|/(2\sqrt{n})) \le 1$. For $I(a, r)$, let

$$f(x) = \max(W(a, |x - a|) - W(a, n), 0).$$

Then $\|f\|_{BMO_{w,p}}$ is independent of I, and

$$|M(f, 0, 1)| \le \left(\int_{I(0,1)} |f|^p \, dx \right)^{1/p}$$

$$\le \left(\int_{|x-a| \le n} \left| \int_{|x-a|}^{n} \frac{w(a, t)^{1/p}}{t^{n/p+1}} \, dt \right|^p dx \right)^{1/p}$$

$$\le C w(a, n)^{1/p} \le C' w(0, n)^{1/p}.$$

This proves (2.4.7). Since $|x - a| \le \sqrt{n} r/2$ for $x \in I(a, r)$, we have

$$M(f, a, r) \ge W(a, \sqrt{n} r/2) - W(a, n)$$

$$= \int_{\sqrt{n} r/2}^{n} \frac{w(a, t)^{1/p}}{t^{n/p+1}} \, dt \ge C \int_{r}^{2\sqrt{n}} \frac{w(a, t)^{1/p}}{t^{n/p+1}} \, dt.$$

This proves (2.4.8).

LEMMA 2.12. *Suppose $f \in bmo_{w,p}(R^n)$ and $g \in L^\infty(R^n)$. Then, fg belongs to $bmo_{w,p}(R^n)$ if and only if*

$$F(f, g) = \sup_I |f_I| \, M_{BMO_{w,p}}(g, I) < \infty.$$

In this case,

$$\text{(2.4.9)} \qquad \big| \|fg\|_{BMO_{w,p}} - F(f,g) \big| \le 2 \|f\|_{BMO_{w,p}} \|g\|_\infty.$$

PROOF. For any cube I, we have

$$\Big| \|(fg)(\cdot) - (fg)_I\|_{L^p(I)} - |f_I| \, \|g(\cdot) - g_I\|_{L^p(I)} \Big|$$

$$\le \|(fg)(\cdot) - (fg)_I - f_I g(\cdot) + f_I g_I\|_{L^p(I)}$$

$$\le \|(f(\cdot) - f_I) g(\cdot)\|_{L^p(I)} + |(fg)_I - f_I g_I| \, |I|^{1/p}$$

$$\left| \frac{1}{|I|} \int_I ((fg)(x) - f_I g(x)) \, dx \right| |I|^{1/p}$$

Hence

$$|MO_{w,p}(fg,I) - f_I\, MO_{w,p}(g,I)| \le 2MO_{w,p}(f,I)\|g\|_\infty$$

which shows (2.4.9).

5. Proof of the theorem.

We write $\Psi(I) = \Psi(a,r)$ for $I = I(a,r)$.

PROOF OF THEOREM 2.1. Suppose, $g \in bmo_{w^*,p}(R^n) \cap L^\infty(R^n)$. For any $f \in bmo_{w,p}(R^n)$ and for any I, by Lemma 2.9, we have

$$|f_I| MO_{w,p}(g,I) \le C\|f\|_{bmo_{w,p}} \Psi(I)^{1/p} MO_{w,p}(g,I)$$
$$\le C\|f\|_{bmo_{w,p}} \|g\|_{BMO_{w^*,p}} < \infty.$$

Therefore, by Lemma 2.12, $fg \in bmo_{w,p}(R^n)$ and

$$\|fg\|_{BMO_{w,p}} \le C\|f\|_{bmo_{w,p}}\|g\|_{BMO_{w^*,p}} + 2\|f\|_{BMO_{w,p}}\|g\|_\infty .$$

Since

$$|M(fg,0,1)| \le \|g\|_\infty (MO(f,0,1) + |M(f,0,1)|),$$

we have

$$\|fg\|_{bmo_{w,p}} \le C(\|g\|_{BMO_{w^*,p}} + \|g\|_\infty)\|f\|_{bmo_{w,p}}$$

which shows that g is a pointwise multiplier on $bmo_{w,p}(R^n)$, and

$$\|g\|_{Op} \le C(\|g\|_{BMO_{w^*,p}} + \|g\|_\infty)$$

where $\|g\|_{Op}$ is the operator norm of g.

Conversely, suppose g is a pointwise multiplier on $bmo_{w,p}(R^n)$. First we show $g \in L^\infty(R^n)$. For any cube $I = I(a,r)$ with $r < 1$, we define $h(x)$ as follows

$$h(x) = \max(W(a,|x-a|) - W(a,r), 0) = \max\left(\int_{|x-a|}^{r} \frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt, 0\right).$$

Then, it follows from Lemma 2.8 and Lemma 2.3 that $\|h\|_{BMO_{w,p}} \le C$ independently of I. For $a > 1 + \sqrt{n}/2$, $M(h,0,1) = 0$, since $I(0,1)$ and the support of h are disjoint. And for $|a| \le 1+\sqrt{n}/2$, by Lemma 2.6,

$$|M(h,0,1)| \le \left(\int_{I(0,1)} |h|^p \, dx\right)^{1/p} \le \left(\int_{|x-a|<1}\left(\int_{|x-a|}^{1} \frac{w(a,t)^{1/p}}{t^{n/p+1}}\, dt\right)^p dx\right)^{1/p} \le Cw(a,1)^{1/p} \le Cw(0,1)^{1/p}.$$

Hence $\|h\|_{bmo_{w,p}} \le C$ independently of I. Now, if $|x-a|<r/2$, then

$$h(x) \ge \int_{r/2}^{r} \frac{w(a,t)^{1/p}}{t^{n/p+1}}\, dt \ge w(a,r/2)^{1/p}\int_{r/2}^{r} \frac{1}{t^{n/p+1}}\, dt \ge C\frac{w(a,r)^{1/p}}{r^{n/p}}.$$

Therefore, by considering the support of h, for $\sigma = M(gh,a,4r)$,

$$\begin{aligned}\int_{I(a,4r)} |gh(x)-\sigma|^p \, dx &\ge \int_{|x-a|<r/2} |gh(x)-\sigma|^p \, dx + \int_{I(a,4r)\setminus I(a,2r)} |\sigma|^p \, dx \\ &\ge \int_{|x-a|<r/2} \left(|gh(x)-\sigma|^p + |\sigma|^p\right) dx \ge \int_{|x-a|<r/2} 2^{1-p}|gh(x)|^p \, dx \\ &\ge C\frac{w(a,r)}{r^n}\int_{|x-a|<r/2} |g(x)|^p \, dx.\end{aligned}$$

Hence

$$\frac{1}{r^n}\int_{|x-a|<r/2} |g(x)|^p \, dx \le \frac{1}{w(a,r)}\int_{I(a,4r)} |gh(x)-\sigma|^p \, dx \le C(\|gh\|_{bmo_{w,p}})^p \le C(\|g\|_{op})^p.$$

Letting r tend to zero, we have

$$|g(a)| \le C\|g\|_{op} \quad a.e. \quad \text{and} \quad \|g\|_{\infty} \le C\|g\|_{op}.$$

Second, we show $g \in bmo_{w,p}(R^n)$. By Lemma 2.12, we have

$$\begin{aligned}\sup_I \|M_{w,p}(g,I)\| &\le \|g\|_{BMO_{w,p}} + 2\|1\|_{BMO_{w,p}}\|g\|_{\infty} \\ &\le (\|g\|_{op} + 2\|g\|_{\infty})\|1\|_{bmo_{w,p}} \le C\|g\|_{op}\|1\|_{bmo_{w,p}}.\end{aligned}$$

for any $f \in bmow,p(Rn)$. Takingf(x) $max(-vV(O,2),-W(O,$ Ix!)), by Lemma 2.10, we have

$$\Psi_1(I)^{1/p} MOw,p(g, I) \leq Glfl\backslash MOw,p(g, I)$$

$$\leq G' \ gllop \ Ifllbmow,p \leq Gil \ Ig \ lop \quad \text{for any } I.$$

And by Lemma 2.11 we have, for any cube I there is a function $f \in bmow,p(Rn)$ such that

$$\Psi_2(I)^{1/p} MOw,p(g, I) \leq Glfl \ MOw,p(g, I)$$

$$\leq G' \ g \ lopl \ fl \ bmow,p \leq Gil \ gllop,$$

where Gil is independent of I and f. These prove $g \in bmow.,p(Rn)$ and

$$|Igi \ BMO_{w^\bullet,p} \leq Gl \ gOp'$$

The proof is complete.

III. Pointwise multipliers on bmo_ϕ, on its subspaces and on hI.

In this chapter, we assume that ϕ is a positive nondecreasing function defined on $\mathbb{R}_+$. $bmo_\phi(\mathbb{R}^n)$ is the function space defined using the mean oscillation and the weight function ϕ. It is a special case of $bmo_{w,p}(R^n)$.

In the first two sections, we state definitions and characterize the pointwise multipliers on $bmo_\phi(\mathbb{R}^n)$. This characterization shows that the pointwise multipliers on $bmo_\phi(\mathbb{R}^n)$ reflect deeply not only the local structure but also the global structure of this space. Therefore we need a weight functon w depended on $a \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$ introduced in the previous chapter. Next we state some sufficient conditions for the pointwise multipliers on $bmo_\phi(\mathbb{R}^n)$, and we give examples of the pointwise multipliers on this space.

In the third section, we consider the pointwise multipliers on subspaces of $bmo_\phi(\mathbb{R}^n)$ by contrast with $bmo_\phi(\mathbb{T}^n)$, on which the pointwise multipliers reflect the only local structure.

In the last section of this chapter, we characterize the pointwise multipliers on the local Hardy space $hI(\mathbb{R}^n)$ introduced by Goldberg [8](1979), whose dual space is a subspace of $bmo_\phi(\mathbb{R}^n)$.

1. Definitions.

For $1 \leq p <$ oo, and for a nondecreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, let $w_{\phi,p}(x, r) = r^n \phi(r)^p$. It follows from John and Nirenberg's inequality and Holder's inequality that

$$bmo_{w_{\phi,p},p}(\mathbb{R}^n) = bmo_{w_{\phi,1},1}(\mathbb{R}^n).$$

Then we define

$$bmo_\phi(\mathbb{R}^n) = bmo_{w_{\phi,p},p}(R^n),$$

$$\Big| \quad {}_{I=I(a,r)} \frac{1}{r \quad r} \int_I |f(x) - f_I \; dx,$$

$$|f_{\,bmo_\phi} = f\|_{BMO_\phi} + M(f,O,l)l\cdot$$

If $\phi(r) = 1$ then $bmo_\phi(\mathbb{R}^n) - bmo(R^n)$, the space of functions of bounded mean oscillation introduced by John and Nirenberg [11](1961). And if $\phi(r) \quad r^\alpha$,

$0 < \alpha \leq 1$, then $bmo_\phi(\mathbb{R}^n)$ coincides with $\Lambda_\alpha(\mathbb{R}^n)$, the space of α-Lipschitz continuous functions (see Meyers [15](1964)). $bmo_\phi(\mathbb{R}^n)$ is a generalization of $bmo(\mathbb{R}^n)$ and $\Lambda_\alpha(\mathbb{R}^n)$.

Let $\Lambda_\phi(\mathbb{R}^n)$ be the set of all measurable functions f such that

$$\operatorname*{ess\,sup}_{x,y\in\mathbb{R}^n} \frac{|f(x) - f(y)|}{\phi(|x - y|)} < \infty.$$

Then

$$\Lambda_\phi(\mathbb{R}^n) \subset bmo_\phi(\mathbb{R}^n).$$

In particular, if $\phi(r)$ 1 then

$$\Lambda_\phi(\mathbb{R}^n) = L^\infty(\mathbb{R}^n) \subsetneq bmo(\mathbb{R}^n) = bmo_\phi(\mathbb{R}^n).$$

If $\phi(r)$ tends to zero as r tends to zero, then any function $f \in \Lambda_\phi(\mathbb{R}^n)$ is continuous. And if $\int_0^1 \phi(t)t^{-1}\,dt < \infty$, then any function $f \in bmo_\phi(\mathbb{R}^n)$ is continuous (see Spanne [26](1965)). Moreover, $\Lambda_\phi(\mathbb{R}^n)$ $bmo_\phi(\mathbb{R}^n)$, if and only if there is a positive constant C such that $\phi(r)^{-1} \int_0^r \phi(t)t^{-1}\,dt \leq C$ for $r > 0$ (see Nakai [20](1984)).

If $\phi(r)/r$ is almost decreasing, i.e. there is a positive constant A such that

(3.1.1) $$\phi(r_2)/r_2 \leq A\phi(r_1)/r_1 \quad \text{for } 0 < r_1 \leq r_2,$$

then

(3.1.2) $$\int_{|x|} \frac{\quad}{t}$$

Therefore, if $\int_r^1 \phi(t)t^{-1}\,dt$ tends to infinity as r tends to zero then $bmo_\phi(\mathbb{R}^n)$ contains not only noncontinuous functions but also unbounded functions (see Lemma 2.8 and [26]).

On the n-dimensional torus $\mathbb{T}^n$, $\Lambda_\phi(\mathbb{T}^n)$ is a subspace of $L^\infty(\mathbb{T}^n)$. Then we have

$$PWM(\Lambda_\phi(\mathbb{T}^n)) = \Lambda_\phi(\mathbb{T}^n).$$

Stegenga [29](1976) has characterized the pointwise multipliers on $bmo(\mathbb{T})$. Using this characterization, he could characterize a class of bounded Toeplitz operators on $H^1(T)$ by use of the fact that the dual space of $H^1(T)$ is $bmo(T)$. This Hardy space H^1, which is a subspace of L^1, is also important function space and has been studied by many aothors.

Janson [10](1976) has characterized the pointwise multipliers on $bmo_\phi(T^n)$ on the assumption that $\phi(r)/r$ is almost decreasing. His characterization is as follows:

$$PWM(bmo_\phi(\mathbf{T}^n)) = bmo_\psi(\mathbf{T}^n) \cap L^\infty(\mathbf{T}^n),$$

where $\psi(r) = \phi(r)/\int_r^1 \phi(t)t^{-1}\,dt$. This characterization shows that the pointwise multipliers on $bmo_\phi(\mathbf{T}^n)$ reflect deeply the local structure.

Moreover, Using the duality, Janson showed that $H^1(T^n)$ and $bmo(T^n)$ have the same pointwise multipliers, i.e.

$$PWM(H^1(\mathbf{T}^n)) = bmo_\psi(\mathbf{T}^n) \cap L^\infty(\mathbf{T}^n),$$

where $\psi(r) = 1/\log(1/r)$.

However, in contrast with that

$$PWM(H^1(R^n)) = \{\text{constant functions}\}.$$

The dual space of $H^1(R^n)$ is $BMO(R^n)$ modulo constants.

Let $h^1(R^n)$ be the local Hardy space introduced by D.Goldberg [8]. Then the dual space of $h^1(R^n)$ is a subspace of $bmo(R^n)$.

We define a subspace of $bmo_\phi(\mathbf{R}^n)$ as follows:

$$ubmbmo_\phi(\mathbf{R}^n) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbf{R}^n) : \|f\|_{BMO_\phi} + \sup_{a \in R^n} |M(f,a,1)| < \infty \right\}.$$

This is a Banach space equipped with a norm

$$\|f\|_{ubmbmo_\phi} = \|f\|_{BMO_\phi} + \sup_{a \in R^n} |M(f,a,1)|.$$

For $\phi(r) = 1$, we denote $ubmbmo_\phi(\mathbf{R}^n)$ by $ubmbmo(R^n)$. Goldberg [8, Corollary 1] introduced $ubmbmo(R^n)$, by the symbol bmo, and show that it is the dual space of $h^1(R^n)$.

2. Pointwise multipliers on $bmo_\phi(\mathbf{R}^n)$.

Now we characterize the pointwise multipliers on $bmo_\phi(\mathbf{R}^n)$. If ϕ satisfies the condition (3.1.1) then, for $r > 0$ and $s \geq 1$,

$$\frac{1}{2^{n+1}A} \leq \frac{r^n \phi(r)}{(2r)^n \phi(2r)} \leq \frac{1}{2^n},$$

$$\int_0 \frac{\qquad}{t} \; t \leq \frac{r}{n} \quad r,$$

$$(sr)^n \phi(sr) \leq A s^{n+1} r^n \phi(r).$$

Therefore, by Theorem 2.1 in the previous chapter, we have the following:

THEOREM 3.1. *If $\phi(r)/r$ is almost decreasing) then*

$$PWM(bmo_\phi(\mathbf{R}^n)) = bmo_{w^*,1}(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n),$$

where

$$w^*(a,r) = r^n \phi(r) \Big/ \left(\left| \int_r^1 \phi(t)\, t^{-1}\, dt \right| + \int_1^{2+|a|} \phi(t)\, t^{-1}\, dt \right). \tag{3.2.1}$$

Moreover, the operator norm of $g \in PWM(bmo_\phi(\mathbf{R}^n))$ is comparable to

$$\|g\|_{BMO_{w^*,1}} + \|g\|_{L^\infty}.$$

This result contrasts with the case of torus. ϕ depends only on $r \in \mathbf{R}_+$. However w^* depends not only on $r \in \mathbf{R}_+$ but also on $a \in \mathbf{R}^n$. The pointwise multipliers on $bmo_\phi(\mathbf{R}^n)$ reflect not only the local structure but also the global structure of this function space.

We note that the assumption "$\phi(r)/r$ is almost decreasing" is able to be replaced by "ϕ is concave". We have learned this from J. Peetre:

LEMMA (Peetre). *If $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ is nondecreasing and $\phi(r)/r$ is almost decreasing, then there is a nondecreasing concave function $\psi : \mathbf{R}_+ \to \mathbf{R}_+$ such that ψ is comparable to ϕ i.e. there is a positive constant C such that*

$$C^{-1} \leq \psi(r)/\phi(r) \leq C \quad \text{for any } r > 0.$$

Next, as consequences of the above theorem, we give some sufficient conditions for the pointwise multipliers on $bmo_\phi(\mathbf{R}^n)$, corresponding to those in the torus case, Stegenga [29, Corollary 2.8].

Let $\phi(r)/r$ be almost decreasing. We define strictly positive functions $\Phi^*(r)$ and $\Phi_*(r)$ as follows:

$$(3.2.2)\qquad \Phi^*(r) = \int_1^{\max(2,r)} \frac{\phi(t)}{t}\,dt \quad \text{and} \quad \Phi_*(r) = \int_{\min\{1,r\}}^{2} \frac{\phi(t)}{t}\,dt.$$

Then, it follows from (3.1.2) and Lemma 2.3 in the previous chapter that:

$$(3.2.3)\qquad \Phi^*(|x|), \quad \Phi_*(|x|) \quad \text{is in } bmo_\phi(\mathbb{R}^n).$$

And $\Phi^*(2r)$ and $\Phi_*(2r)$ are comparable to $\Phi^*(r)$ and $\Phi_*(r)$, respectively, i.e. there are positive constants C_1 and C_2 such that

$$(3.2.4)\qquad \Phi^*(r) \le \Phi^*(2r) \le C_1\Phi^*(r), \quad \Phi_*(r) \ge \Phi_*(2r) \ge C_2\Phi_*(r) \quad \text{for } r > 0.$$

Therefore w^* in (3.2.1) is comparable to

$$\frac{r^n\phi(r)}{\Phi_*(r) + \Phi^*(r) + \Phi^*(|a|)}.$$

With definition (3.2.2), we state the following:

PROPOSITION 3.2. *Let g be a measurable function. If there are constants $C_1 > 0$, $C_2 > 0$ and $c \in \mathbb{C}$ such that, for $x, y \in \mathbb{R}^n$ with $|y| \le 1$,*

$$(3.2.5)\qquad | \quad < \frac{C_1\phi(|y|)}{\quad} \quad |$$

$$(3.2.6)\qquad | \quad \frac{C_2}{|\quad|}$$

where $\phi(0+) = \lim_{r\downarrow 0}\phi(r)$, then g is a pointwise multiplier on $bmo_\phi(\mathbb{R}^n)$.

COROLLARY 3.3. *Let $g = g_1/g_2$. If there are positive constants C_i $(i = 1, 2, 3, 4)$ such that, for $x, y \in \mathbb{R}^n$,*

$$|g_1(x)| \le C_1, \quad \text{and} \quad |g_1(x) - g_1(y)| \le C_2|x - y|,$$

$$|g_2(x)| \ge C_3\Phi^*(|x|) \quad \text{and} \quad |g_2(x) - g_2(y)| \le C_4|x - y|,$$

then g is a pointwise multiplier on $bmo_\phi(\mathbb{R}^n)$.

We prove these proposition and corollary, by using the property (3.2.4).

PROOF OF PROPOSITION 3.2. We may assume that c = 0, since constant functions are clearly in $PWM(bmo_\phi(\mathbb{R}^n))$. It follows from (3.2.6) that 9 E Loo(Rn). Next we show that, for any cube I I(a, r),

(3.2.7) $$\frac{\quad|}{\phi(r)} \quad g, \quad \leq \quad .$$

Case 1: $r \leq 1/\sqrt{n}$ and $\phi(0+) = 0$. By Lemma 2.2 and (3.2.5), we have

$$\frac{2}{|} \int |g(x) - g(a)|\, dx < \frac{2C_1\phi(r)}{}$$

Hence (3.2.7) follows.

Case 2: $r \leq 11\sqrt{n}$ and $\phi(0+) > 0$. We have

$$MO(g, I) \leq \frac{2}{|I|} \Big| \qquad | \qquad \frac{2C_1\phi\ r}{}$$

And, since $\Phi^*(|x|)$ is comparable to $\Phi^*(|a|)$ for x E I, by (3.2.4), we have

$$|\ | \qquad \frac{C}{|} \qquad \frac{C\phi(r)}{|}$$

Hence (3.2.7) follows.

Case 3: $11\sqrt{n} \leq r \leq |a|/\sqrt{n}$. If x E I, then lal/2 ≤ |x| ≤ 3\aI/2. Therefore $\Phi^*($ x $)$ is comparable to $\Phi^*(|a$ $)$ for x E I. Then we have

$$| \qquad \frac{C}{|} \qquad \frac{C\phi(r)}{|}$$

Hence (3.2.7) follows.

Case 4: $11\sqrt{n} \leq r$ and $a\ /\sqrt{n} \leq r$. If x E I, then $x \leq 2nr$. By using the inequality;

$$\int_{e^2}^{r} \frac{1}{\log p}\, dp < \frac{2r}{\log r} - e^2 \quad \text{for } r \geq e^2,$$

we have

$$|\ | \qquad C \qquad \qquad 1 \quad dx$$

$$< \frac{C'}{r} \int_0^{2nr} \frac{1}{\Phi^*(p)}\, dp = \frac{C'}{r} \int_0^{2nr} \frac{1}{\ 1 \qquad t\ } d$$

$$< \frac{C'}{r\phi(1)} \int_0^{2nr} \frac{1}{\log \max(2, p)}\, dp < \frac{C''}{\log 2nr}.$$

On the other hand,

$$\Phi^*(r) \leq \phi(\max(2,r)) \log \max(2,r) \leq C\phi(r) \log 2nr.$$

Hence (3.2.7) follows.

PROOF OF COROLLARY 3.3. It follows form the assumption that

$$\left| 9 \quad Y \quad 9 \quad - \right| \frac{}{92(X+Y)92(X)} \Bigg|$$

$$\leq \frac{|91(X+Y)\ (92(X+Y)-92(X))I+\ (91(X+Y)-91(X))192(X+Y)|}{|92(x+y)|\,|92(x)|}$$

$$\leq C \frac{|Y|}{\Phi^*(|x|)}.$$

By the almost decreasingness of $\phi(r)/r$, we have

$$\Phi_*(r) = \int_r^2 \frac{\phi(t)}{t}\,dt \leq \int_r^2 A\frac{\phi(r)}{r}\,dt \leq 2A\frac{\phi(r)}{r} \quad \text{for } r \leq 1.$$

Hence

$$, \quad \frac{\phi(|y|)}{|\ |\quad |} \quad <'' \quad \frac{\phi(|y|)}{|\ |} \quad .$$

Therefore we have (3.2.5). And, by the boundedness of 91 and 1/92, we have (3.2.6).

At the end of this section, we give examples of the pointwise multiplers on $bmo_\phi(\mathbb{R}^n)$.

EXAMPLES 1. By Corollary 3.3, the following are pointwise multipliers on $bmo_\phi(\mathbb{R}^n)$:

$$\frac{1}{\Phi^*(|x|)}, \quad \frac{\sin x}{\Phi^*(|x|)}, \quad \frac{1}{1+|x|}, \quad \frac{\sin \Phi^*(|x|)}{1+|x|}.$$

EXAMPLES 2. Assume that $1 \leq \alpha$ and $\beta \leq \alpha$, or assume that $1 \leq \alpha < \beta \leq \alpha + 1$ and $\phi(0+) > 0$. By Proposition 3.2, the following are pointwise multipliers on $bmo_\phi(\mathbb{R}^n)$:

$$\frac{\sin |x|^\beta}{(1+|x|)^\alpha}, \quad \frac{\sin \Phi^*(|x|)^\beta}{\Phi^*(|x|)^\alpha}.$$

Janson [10, p.196] has given a pointwise multiplier on $bmo(T^n)$ which is not continuous. We also construct a pointwise multiplier on $bmo_\phi(\mathbb{R}^n)$ which is not continuous.

EXAMPLES 3. Let

$$- \frac{}{\Phi_*(r)} \quad \text{and} \quad \Psi_*(r) = \int_{\min(1,r)}^{2} \frac{\psi(t)}{t}\, dt.$$

If $\psi(t)/t$ is almost decreasing then the following is a pointwise multiplier on $bmo_\phi(\mathbb{R}^n)$:

$$g(x) = \frac{|\;|}{\Phi^*(|x|)}.$$

In fact, $g(x)$ Csin $\Psi_*(|x|)$ for $|x| \le 2$, and $g(x) = C'/\Phi^*(|x|)$ for $|x| \ge 1$. Since ψ satisfies the condition (3.1.1), $\Psi_*(|x|)$ is in $bmo_\psi(\mathbb{R}^n)$, and so $\sin \Psi_*(|x|)$ is in $bmo_\psi(\mathbb{R}^n)$ by Lemma 2.3. $C'/\Phi^*(|x|)$ is a pointwise multiplier on $bmo_\phi(\mathbb{R}^n)$ by Example 1. Hence, for $r \le 1/\sqrt{n}$, we have the inequality (3.2.7). And, for $r \ge 1/\sqrt{n}$, in a way similar to the cases 3 and 4 in Proof of Proposition 3.2, we have the inequality (3.2.7). Then g is a pointwise multiplier on $bmo_\phi(\mathbb{R}^n)$. Moreover, if $\Psi^*(r)$ tends to infinity as r tends to zero, then g is not continuous.

3. Pointwise multipliers on subspaces of $bmo_\phi(\mathbb{R}^n)$.

In this section, we consider the pointwise multipliers on $bmo_\phi(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ and on $ubmbmo_\phi(\mathbb{R}^n)$, which are subspaces of $bmo_\phi(\mathbb{R}^n)$. In these cases, we have results similar to the torus case, i.e. the pointwise multipliers reflect the only local structure of these spaces.

$bmo_\phi(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ is a Banach space equipped with a norm $\|\cdot\|_{bmo_\phi} + \|\cdot\|_{L^p}$. And this space has the generalized Chebyshev's inequality (1.1.4). Therefore, in a way similar to Lemmas 1 and 2 in Chapter I, we have that any pointwise multiplier from $bmo_\phi(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ to $bmo_\phi(\mathbb{R}^n)$ is a bounded operator.

And $ubmbmo_\phi(\mathbb{R}^n)$ is also a Banach space and has the generalized Chebyshev's inequality. Then any pointwise multiplier from $ubmbmo_\phi(\mathbb{R}^n)$ to $bmo_\phi(\mathbb{R}^n)$ is a bounded operator.

We have the following theorems similar to the torus case.

THEOREM 3.4. *Suppose that $\phi(r)/r$ is almost decreasing.*

(i) *Let $1 \le p < \infty$. Then a function g is a pointwise multiplier from $bmo_\phi(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ to $bmo_\phi(\mathbb{R}^n)$ if and only if g is in $bmo_\psi(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.*

where

$$\psi(r) = \phi(r) \Big/ \int_{\min(1,r)}^{2} \frac{\phi(t)}{t}\, dt\,. \tag{3.3.1}$$

In this case,

$$PWM(bmo_\phi(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)) = bmo_\psi(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n),$$

and the operator norm of $g \in PWM(bmo_\phi(\mathbb{R}^n) \cap L^p(\mathbb{R}^n))$ *is comparable to*

$$\|g\|_{BMO_\psi} + \|g\|_{L^\infty}.$$

(ii) *A function* g *is a pointwise multiplier from* $bmo_\phi(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ *to* $bmo_\phi(\mathbb{R}^n)$ *if and only if* g *is in* $bmo_\phi(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. *In this case,*

$$PWM(bmo_\phi(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)) = bmo_\phi(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n),$$

and the operator norm of $g \in PWM(bmo_\phi(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$ *is comparable to*

$$\|g\|_{BMO_\phi} + \|g\|_{L^\infty}.$$

THEOREM 3.5. *Suppose that* $\phi(r)/r$ *is almost decreasing. A function* g *is a pointwise multiplier from* $ubmbmo_\phi(\mathbb{R}^n)$ *to* $bmo_\phi(\mathbb{R}^n)$ *if and only if* g *is in* $bmo_\psi(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, *where* ψ *is defined by* (3.3.1). *In this case,*

$$PWM(ubmbmo_\phi(\mathbb{R}^n)) = bmo_\psi(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n),$$

and the operator norm of $g \in PWM(ubmbmo_\phi(\mathbb{R}^n))$ *is comparable to*

$$\|g\|_{BMO_\psi} + \|g\|_{L^\infty}.$$

To prove these theorems, we show the following Lemmas:

LEMMA 3.6. *Let* $1 \le p \le \infty$. *Then there is a constant* $C > 0$ *such that, for any* $f \in bmo_\phi(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ *and for any cube* $I(a,r)$,

$$|M(f,a,r)| \le C(\|f\|_{BMO_\phi} + \|f\|_{L^p})\Phi_*(r). \tag{3.3.2}$$

PROOF. If $p = \infty$, then (3.3.2) is clear. We assume that $1 \le p < \infty$. Let $I = I(a,r)$. For $r \ge 1$, it follows from Holder's inequality that

$$|M(f,a,r)| \le \left|\frac{1}{|I|}\int_I |f(x)|^p\,dx\right|^{1/p} \le \|f\|_{L^p}.$$

And for $r < 1$, it follows from Lemma 2.5 that

$$\begin{aligned}|M(f,a,r)| &\le |M(f,a,r) - M(f,a,1)| + |M(f,a,1)| \\ &\le C\int_r^1 \frac{\phi(t)}{t}\,dt\,\|f\|_{BMO_\phi} + \|f\|_{L^p} \\ &\le C'\int_r^1 \frac{\phi(t)}{t}\,dt\,\left(\|f\|_{BMO_\phi} + \|f\|_{L^p}\right).\end{aligned}$$

Hence (3.3.2) follows.

LEMMA 3.7. *There is a constant $C > 0$ such that, for any $f \in ubmbmo_\phi(\mathbb{R}^n)$ and for any cube $I(a,r)$,*

$$|M(f,a,r)| \le C\|f\|_{ubmbmo_\phi}\Phi_*(r). \tag{3.3.3}$$

PROOF. For $r \ge 1$, let j be the smallest integer satisfying $r \le 2^j$. Then

$$\begin{aligned}M(|f|,a,r) &\le \frac{1}{r^n}\int_{I(a,2^j)} |f(x)|\,dx \le \frac{2^{jn}}{r^n}\sup_{b\in\mathbb{R}^n}\int_{I(b,1)} |f(x)|\,dx \\ &\le 2^n\left(\sup_{b\in\mathbb{R}^n} MO(f,b,1) + \sup_{b\in\mathbb{R}^n}|M(f,b,1)|\right).\end{aligned} \tag{3.3.4}$$

Hence we have

$$|M(f,a,r)| \le C\|f\|_{ubmbmo_\phi}.$$

And for $r \le 1$, we have by Lemma 2.5,

$$|M(f,a,r) - M(f,a,1)| \le C\int_r^2 \frac{MO(f,a,t)}{t}\,dt \le C\int_r^2 \frac{\phi(t)}{t}\,dt\,\|f\|_{BMO_\phi},$$

and

$$|M(f,a,r)| \le C\|f\|_{ubmbmo_\phi}\Phi_*(r).$$

Therefore, we have (3.3.3).

The following lemma shows that the estimate (3.3.2) is sharp.

LEMMA 3.8. *Let* $1 \le p < \infty$. *For any* $a \in \mathbb{R}^n$, *let*

$$f_a(x) = \Phi_*(|x-a|) - \Phi_*(1).$$

Then f_a *is in* $bmo_\phi(\mathbb{R}^n) \cap L^p(\mathbb{R}^n))$, *and*

$$\|f_a\|_{BMO_\phi} + \|f_a\|_{L^p} \le C_1, \tag{3.3.5}$$

$$M(f_a, a, r) \ge C_2 \Phi_*(r) \quad \text{for } r \le 1, \tag{3.3.6}$$

where $C_1 > 0$ *and* $C_2 > 0$ *are independent of* $a \in \mathbb{R}^n$ *and* $r \in \mathbb{R}_+$.

PROOF. By (3.2.3) and Lemma 2.3, we have $f \in bmo_\phi(\mathbb{R}^n)$ and $\|f_a\|_{BMO_\phi} \le C$ independently of $a \in \mathbb{R}^n$. Since the support of f_a is included in $\{x : |x-a| \le 1\}$ and $|f(x)| \le \phi(1) \log |x-a|$, we have $f \in L^p(\mathbb{R}^n)$ and $\|f_a\|_{L^p} \le C_p$, where the positive constant C_p is dependent on p and independent of $a \in \mathbb{R}^n$. Hence we have (3.3.5). Next, we show (3.3.6). Since $\Phi_*(r)$ is decreasing for $r \le 1$, we have

$$f_a(x) \ge \Phi_*(r/2) - \Phi_*(1) \ge C\Phi_*(r/2) \ge C'\Phi_*(r) \quad \text{for } |x-a| \le r/2,\ r \le 1.$$

Hence

$$M(f_a, a, r) \ge \frac{1}{r^n} \int_{\{|x-a| \le r/2\}} f_a(x)\, dx \ge C''\Phi_*(r) \quad \text{for } r \le 1.$$

LEMMA 3.9. *Let* $1 \le p \le \infty$. *If* g *is a pointwise multiplier from* $bmo_\phi(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ *to* $bmo_\phi(\mathbb{R}^n)$, *then* g *is in* $L^\infty(\mathbb{R}^n)$ *and* $\|g\|_{L^\infty} \le C\|g\|_{Op}$ *where* $C > 0$ *is independent of* g.

PROOF. For any cube $I = I(a, r)$ with $r \le 1$, we define a function $h \in bmo_\phi(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ as follows:

$$h(x) = \begin{cases} \exp(i\Phi^*(|x-a|)) - \exp(i\Phi^*(r)) & |x-a| < r \\ 0 & r \le |x-a|. \end{cases}$$

Then, by Lemma 2.3, we have $\|h\|_{BMO_\phi} \le C_0 \|\Phi(\cdot)\|_{BMO_\phi}$. And, since $|h(x)| \le 2$ and the support of h is included in $I(a, 2)$, we have $\|h\|_{L^p} \le C_{p_0}$. Since g is a bounded operator, we have

$$\|gh\|_{bmo_\phi} \le \|g\|_{Op}(\|h\|_{BMO_\phi} + \|h\|_{L^p}) \le C\|g\|_{Op},$$

independently of I. It follows that

(3.3.7) $$MO(gh,a,4r) \leq C_1 \|g\|_{\mathcal{O}_p}\phi(4r).$$

Let η_1 and η_2 be constants such that $\log\eta_1 = \pi/\phi(1)$ and $1 < \eta_2 < \eta_1$. And let

$$L_r = \{x \in \mathbb{R}^n : r/\eta_1 \leq |x-a| \leq r/\eta_2\}.$$

Then we have, for $x \in L_r$,

$$\Phi_*(|x-a|) - \Phi_*(r) \leq \int_{r/\eta_1}^{r} \frac{\phi(t)}{t}\,dt \leq \pi \quad \text{and}$$

$$\Phi_*(|x-a|) - \Phi_*(r) \geq \int_{r/\eta_2}^{r} \frac{\phi(t)}{t}\,dt \geq C\phi(r),$$

since $\phi(r)/r$ is almost decreasing. So the inequality

$$|e^{i\theta}-1| \geq \frac{2\theta}{\pi} \quad \text{for } 0 \leq \theta \leq \pi$$

implies that $h(x) \geq C'\phi(r)$ for $x \in L_r$. Let $\sigma = M(gh,a,4r)$. Then we have, by considering the support of h,

(3.3.8) $$MO(gh,a,4r) = \int_{I(a,4r)} |gh(x)-\sigma|\,dx$$
$$\geq \int_{L_r} |gh(x)-\sigma|\,dx + \int_{I(a,4r)\setminus I(a,2r)} |\sigma|\,dx \geq \int_{L_r} (|gh(x)-\sigma| + |\sigma|)\,dx$$
$$\geq \int_{L_r} |gh(x)|\,dx \geq C'\phi(r)\int_{L_r} |g(x)|\,dx.$$

From (3.3.7) and (3.3.8) it follows that

$$\frac{1}{|L_r|}\int_{L_r} |g(x)|\,dx \leq C'' \|g\|_{\mathcal{O}_p}.$$

Letting r tend to zero, we have

$$|g(a)| \leq C''\|g\|_{\mathcal{O}_p} \quad \text{a.e.} \quad \text{and} \quad \|g\|_{L^\infty} \leq C''\|g\|_{\mathcal{O}_p}.$$

Now we prove the theorems.

PROOF OF THEOREM 3.4. (i) Case $1 \le P < oo$. Suppose that 9 is in $bmo_\psi(\mathbb{R}^n)$ n *Loo(Rn).* For any $I = I(a, r)$ and for any I E $bmo_\phi(\mathbb{R}^n)$ n *LP(Rn),* by Lemma 3.6, we have

$$\Big| \frac{MO(g,I)}{} \qquad \Phi_*(r)$$

$$\le C'(I \ {}_{BMO_\phi} + II \ LP) \ 9 \ {}_{BMO_\psi}.$$

Therefore, by Lemma 2.12, we have

$$II \, 9 \ {}_{BMO_\phi} \le C(I \ {}_{BMO_\phi} + I \ LP) \ 9 \ {}_{BMO_\psi} + 2 \ I \ {}_{BMO_\phi} \ 9 \ \mathrm{Loo},$$

which shows that 9 is a pointwise multiplier from $bmo_\phi(\mathbb{R}^n)$ n *LP(Rn)* to $bmo_\phi(\mathbb{R}^n)$, and

$$Ig I \ Op \le C(I \ gl \ {}_{BMO_\psi} + IIg \ \mathrm{Loo}).$$

Conversely, suppose that 9 is a pointwise multiplier from $bmo_\phi(\mathbb{R}^n)$ n *LP(Rn)* to $bmo_\phi(\mathbb{R}^n)$. By Lemma 3.9, we have 9 E *Loo(Rn)* and 19 Loo $\le$ CI 9 *Iop.* From Lemma 2.12 it follows that

$$\sup_{I=I(a,r)} \Big| \qquad \Big| \quad \Big| \qquad \Big| \quad \Big| \qquad \Big| \ \Big|$$

$$\le C\|g\|_{Op}(\|f\|_{BMO_\phi} + \|f\|_{L^p}).$$

Taking $I = Ia$ in Lemma 3.8, we have,

$$\sup_{r\le 1, a\in\mathbb{R}^n} \frac{\Phi_*(r)}{} MO(g,a,r) \le C' \, |9 \ op.$$

And, for $r > 1$, we have

$$\frac{\Phi_*(r)}{\phi(r)} MO(g,a,r) \le 2\frac{\Phi_*(1)}{\phi(1)}\|g\|_{L^\infty} \le C''\|g\|_{Op}.$$

Hence g is in $bmo_\psi(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and

$$I9 \ {}_{BMO_\psi} + IgLoo \le C \ 9 \ op.$$

(ii) Case $p = oo$. Suppose that 9 is in $bmo_\phi(\mathbb{R}^n)$ n *Loo(Rn).* For any cube $1 = I(a, r)$, we have

$$\Big| \ I \qquad \phi(r) \quad - \qquad \mathrm{Loo} \qquad \phi(r) \quad - \qquad \mathrm{Loo} \quad g\|_{BMO_\phi}.$$

Therefore, by Lemma 2.12, we have

$$\|fg\|_{BMO_\phi} \le C\|f\|_{L^\infty}\|g\|_{BMO_\phi} + 2\|f\|_{BMO_\phi}\|g\|_{L^\infty}$$

which shows that g is a pointwise multiplier from $bmo_\phi(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ to $bmo_\phi(\mathbb{R}^n)$, and

$$\|g\|_{Op} \le C(\|g\|_{BMO_\phi} + \|g\|_{L^\infty}).$$

Conversely, suppose that g is a pointwise multiplier from $bmo_\phi(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ to $bmo_\phi(\mathbb{R}^n)$. By Lemma 3.9, we have $g\in L^\infty(\mathbb{R}^n)$ and $\|g\|_{L^\infty}\le C\|g\|_{Op}$. Since $1\in bmo_\phi(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, g is in $bmo_\phi(\mathbb{R}^n)$ and $\|g\|_{BMO_\phi}\le C\|g\|_{Op}$.

PROOF OF THEOREM 3.5. Suppose that g is in $bmo_\psi(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$. For any $f\in ubmbmo_\phi(\mathbb{R}^n)$ and for any cube $I = I(a,r)$, by Lemma 3.7, we have

$$\Big| \ \frac{\quad}{\phi(r)} \ \Big| \ \frac{\Phi_*(r)}{\quad}$$

$$\le C\|f\|_{ubmbmo_\phi}\|g\|_{BMO_\psi}.$$

From Lemma 2.12 it follows that

$$\|fg\|_{BMO_\phi} \le C\|f\|_{ubmbmo_\phi}\|g\|_{BMO_\psi} + 2\|f\|_{BMO_\phi}\|g\|_{L^\infty}$$

which shows that g is a pointwise multiplier from $ubmbmo_\phi(\mathbb{R}^n)$ to $bmo_\phi(\mathbb{R}^n)$, and

$$\|g\|_{Op} \le C(\|g\|_{BMO_\psi} + \|g\|_{L^\infty}).$$

Conversely, suppose that g is a pointwise multiplier from $ubmbmo_\phi(\mathbb{R}^n)$ to $bmo_\phi(\mathbb{R}^n)$. Since $bmo_\phi(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\subset ubmbmo_\phi(\mathbb{R}^n)$, by Theorem 3.4, we have that g is in $bmo_\psi(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ and

$$\|g\|_{BMO_\psi} + \|g\|_{L^\infty} \le C\|g\|_{Op}.$$

Next, we give a sufficient condition for the pointwise multipliers on $bmo_\phi(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$.

PROPOSITION 3.10. *Suppose that $\phi(r)/r$ is almost decreasing. And suppose that g is bounded and that there is a constant $C > 0$ such that*

$$\Big| \ \frac{\quad}{|\ |} \ \quad \text{for } x, y\in\mathbb{R}^n \text{ with } |y|\le 1,$$

where Φ_* is defined by (3.2.2). *Then g is a pointwise multiplier from $bmo_\phi(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ to $bmo_\phi(\mathbb{R}^n)$ $(1 \leq p \leq \infty)$, and from $ubmbmo_\phi(\mathbb{R}^n)$ to $bmo_\phi(\mathbb{R}^n)$.*

PROOF. For $r \leq 1/\sqrt{n}$, we have

$$MO(g, I) \leq \frac{2}{|I|} \quad 2C\phi(r)$$

And, for $r \geq 1/\sqrt{n}$, we heve

$$MO(g, I) \leq 2 \|g\|_{L^\infty} \quad \text{and} \quad C \leq \frac{\phi(r)}{\Phi_*(r)}.$$

Hence g is in $bmo_\psi(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

4. Pointwise multipliers on $h^1(\mathbb{R}^n)$.

The end of this chapter, we characterize the pointwise multipliers on the local Hardy space $h^1(\mathbb{R}^n)$.

By (3.3.4), $\|f\|_{ubmbmo}$ is equivalent to

(3.4.1) $$\sup_{r<1, a\in\mathbb{R}^n} MO(f, a, r) + \sup_{r\geq 1, a\in\mathbb{R}^n} M(|f|, a, r).$$

Goldberg [8, Corollary 1] introduced $ubmbmo(\mathbb{R}^n)$, using (3.4.1), by the symbol *bmo*, and showed that it is the dual space of $h^1(\mathbb{R}^n)$.

Recall the properties of $h^1(\mathbb{R}^n)$. Let R_j be the j-th Riesz transform given by

$$\widehat{R_j f} = \frac{i\xi_j}{|\xi|}\hat{f},$$

where $\hat{f}$ is the Fourier transform of f. Fix $u \in C_0^\infty(\mathbb{R}^n)$ satisfying $u = 1$ in a neighborhood of the origin, and define

$$\widehat{r_j f} = u\,\frac{i\xi_j}{|\xi|}\hat{f}.$$

Then $f \in h^1(\mathbb{R}^n)$ if and only if $f, r_1 f, \ldots, r_n f \in L^1(\mathbb{R}^n)$ (see [8, Theorem 2]). Also as in the proof of Corollary 1 in [8], we have

(3.4.2) $$\|r_j R_k f\|_{ubmbmo} \leq C\|f\|_{L^\infty} \quad \text{for } f \in L^\infty(\mathbb{R}^n).$$

THEOREM 3.6. *Let*

$$\psi(r) = 1 \Big/ \int_{\min(1,r)}^{2} t^{-1}\, dt .$$

Then

$$PWM(h^1(\mathbb{R}^n)) = bmo_\psi(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n),$$

and the operator norm of $g \in PWM(h^1(\mathbb{R}^n))$ is comparable to

$$\|g\|_{BMO_\psi} + \|g\|_{L^\infty} .$$

PROOF. Suppose that g is a pointwise multiplier on $h^1(\mathbb{R}^n)$. By the duality and by Theorem 3.5, g is in $bmo_\psi(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and the operator norm of $g \in PWM(h^1(\mathbb{R}^n))$ is comparable to $\|g\|_{BMO_\psi} + \|g\|_{L^\infty}$.

Conversely, suppose that g is in $bmo_\psi(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Let $h \in h^1(\mathbb{R}^n)$. Then, for any $f \in C_0^\infty(\mathbb{R}^n)$, by the duality, Theorem 3.5 and (3.4.2), we have

$$\left| \int (r_j R_k f) g h \, dx \right| \le C \|f\|_{L^\infty} \|h\|_{h^1} \quad \text{for } j, k = 1, \ldots , n.$$

Hence $R_k r_j (gh)$ is a bounded measure on $\mathbb{R}^n$. Thus, by the n-dimensional F. and M. Riesz theorem, $r_j(gh) \in L^1(\mathbb{R}^n)$ $(j = 1, \ldots , n)$. Since $gh \in L^1(\mathbb{R}^n)$ is clear, we have $gh \in h^1(\mathbb{R}^n)$. Therefore g is a pointwise multiplier on $h^1(\mathbb{R}^n)$.

IV. Pointwise multipliers on Morrey spaces and on the spaces of functions of bounded mean oscillation with the Muckenhoupt weight.

In this chapter, as consequences of Theorem 2.1 in Chapter II, we characterize the pointwise multipliers on Morrey spaces and on the spaces of functions of bounded mean oscillation with the Muckenhoupt weight. One of the latter is a generalized Morrey space.

1. Pointwise multipliers on Morrey spaces.

For $1 \leq p < \infty$, $0 \leq \lambda < n$ and $w(x,r) = r^\lambda$, $bmo_{w,p}(\mathbb{R}^n)$ is the Morrey space. We characterize the pointwise multipliers on the Morrey space.

THEOREM 4.1. *Let* $1 \leq p < \infty$, $0 \leq \lambda < n$ *and* $w(x,r) = r^\lambda$. *Then*

$$PWM(bmo_{w,p}(\mathbb{R}^n)) = bmo_{w,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n).$$

Moreover, the operator norm of $g \in PWM(bmo_{w,p}(\mathbb{R}^n))$ *is comparable to*

$$\|g\|_{bmo_{w,p}} + \|g\|_{L^\infty}.$$

PROOF. Case 1: $0 < \lambda < n$. Since w satisfies from (2.2.1) to (2.2.4), we have by Theorem 2.1

$$PWM(bmo_{w,p}(\mathbb{R}^n)) = bmo_{w^*,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n),$$

where $w^* = w/\Phi$, $\Phi = \Phi_1 + \Phi_2$ and

$$\Psi_1(a,r) = \left| \frac{p}{n-\lambda} \left| 1 - \max(2,|a|,r)^{-(n-\lambda)/p} \right| \right|^p \quad \text{and}$$

$$\Psi_2(a,r) = \left| \frac{p}{n-\lambda} \left| r^{-(n-\lambda)/p} - \max(2,|a|,r)^{-(n-\lambda)/p} \right| \right|^p.$$

$\Psi_1(a,r)$ is comparable to 1, $\Psi_2(a,r)$ is comparable to $r^{-(n-\lambda)}$ for $r \leq 1$, and $\Psi_2(a,r)$ is less than a constant for $r > 1$. Hence,

$$w^*(a,r) \quad \text{is comparable to} \quad \begin{cases} r^n & r \leq 1 \\ w(a,r) = r^\lambda & r > 1. \end{cases}$$

Since $MO_{w^*,p}(g,a,r) \le C\|g\|_\infty$ for $r \le 1$, we have

$$bmo_{w^*,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) = bmo_{w,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$$

and

$$\|g\|_{BMO_{w^*,p}} + \|g\|_{L^\infty} \text{ is comparable to } \|g\|_{BMO_{w,p}} + \|g\|_{L^\infty}.$$

Case 2: λ 0 then

$$bmo_{w,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n) + \{\text{constants}\},$$

i.e. for any $f \in bmo_{w,p}(\mathbb{R}^n)$, there are $f_p \in L^p(\mathbb{R}^n)$ and $c_f \in \mathbb{C}$ such that $f = f_p + c_f$, and $\|f\|_{bmo_{w,p}}$ is comperable to $\|f_p\|_{L^p} + |c_f|$. Therefore

$$PWM(bmo_{w,p}(\mathbb{R}^n)) = L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n),$$

$$\|g\|_{Op} \text{ is comparable to } \|g\|_{L^p} + \|g\|_{L^\infty}.$$

The Proof is complete.

Next we define, for $0 < p < \infty$ and $0 \le \lambda < n$,

$$M_{p,\lambda}(\mathbb{R}^n) = \left\{ f : \sup_{I=I(a,r)} \frac{1}{r^\lambda} \int_I |f(x)|^p \, dx < \infty \right\},$$

$$\|f\|_{M_{p,\lambda}} = \sup_{I=I(a,r)} \left(\frac{1}{r^\lambda} \int_I |f(x)|^p \, dx \right)^{1/p}.$$

This is also called the Morrey space. $M_{p,\lambda}(\mathbb{R}^n)$ is a Banach space for $1 \le p < \infty$ and a complete quasi-normed linear space for $0 < p < 1$, and satisfies the condition (1.1.2) in Chapter I. Moreover $M_{p,\lambda}(\mathbb{R}^n)$ includes $L^p_{\text{comp}}(\mathbb{R}^n)$. From Theorem 1.1 it follows that

$$PWM(M_{p,\lambda}(\mathbb{R}^n)) = L^\infty(\mathbb{R}^n) \quad \text{and} \quad \|g\|_{Op} = \|g\|_{L^\infty},$$

where $\|g\|_{Op}$ is the operator norm of $g \in PWM(M_{p,\lambda}(\mathbb{R}^n))$.

On the torus,

$$bmo_{w,p}(\mathbb{T}^n) = M_{p,\lambda}(\mathbb{T}^n).$$

Then, in contrast with the case of $\mathbb{R}^n$, we have

$$PWM(bmo_{w,p}(\mathbf{T}^n)) = L^\infty(\mathbf{T}^n).$$

2. Pointwise multipliers on the spaces of functions of bounded mean oscillation with the Muckenhoupt weight.

We recall the definitions of the classes of weights A_p (see Muckenhoupt [17] and [18]). A locally integrable and nonnegative function u is said to belong to A_p, $1 < p < \infty$, if there is a constant C such that

$$\left(\frac{1}{|I|}\int_I u(x)dx\right)\left(\frac{1}{|I|}\int_I u(x)^{-1/(p-1)}\,dx\right)^{p-1} \le C$$

for any I, and is said to belong to A_1, if there is a constant C such that

$$\frac{1}{|I|}\int_I u(x)\,dx \le C \operatorname*{essinf}_I u$$

for any I.

THEOREM 4.2. *Let $1 \le p < \infty$, $0 < \alpha \le \min(p, (n+p)/n)$, $1 \le q \le (n+p)/n\alpha$ and*

$$w(I) = \left(\int_I u(x)\,dx\right), \qquad u \in A_q.$$

Then

$$PWM(bmo_{w,p}(\mathbb{R}^n)) = bmo_{w^*,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n),$$

where $w^* = w/\Phi$, $\Phi = \Phi_1 + \Phi_2$ and

(4.2.1) $$\Psi_1(a,r) = \left(\int_{I(0,\max(2,|a|,r))\setminus I(0,1)} u(x)^{\alpha/p}|x|^{-n(1-\alpha/p+1/p)}\,dx\right)^p,$$

(4.2.2) $$\Psi_2(a,r) = \left(\int_{I(a,\max(2,|a|,r))\setminus I(a,r)} u(x)^{\alpha/p}|x-a|^{-n(1-\alpha/p+1/p)}\,dx\right)^p.$$

Moreover, the operator norm of $g \in PWM(bmo_{w,p}(\mathbb{R}^n))$ is comparable to

$$\|g\|_{bmo_{w^*,p}} + \|g\|_{L^\infty}.$$

THEOREM 4.3. *Let* $1 \le p < \infty$, $0 < \alpha < 1$, $1 \le q \le 1/\alpha$ *and*

$$w(I) = \left| \int_I u(x)\, dx \right|^{\alpha}, \qquad u \in A_q.$$

Then

$$PWM(bmo_{w,p}(\mathbb{R}^n)) = bmo_{w^*,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n),$$

where

$$\overset{*}{w}(a,r) = \frac{w(a,r)}{\qquad} \tag{4.2.3}$$

Moreover, the operator norm of $g \in PWM(bmo_{w,p}(\mathbb{R}^n))$ *is comparable to*

$$\|g\|_{bmo_{w^*,p}} + \|g\|_{L^\infty}.$$

In the Theorem 4.3, $bmo_{w,p}(\mathbb{R}^n)$ is a generalized Morrey space.

To prove these theorems, we state some basic properties of A_p weights. (See for example [7].)

LEMMA 4.4. *If u belongs to A_p, $1 \le p < \infty$, then there are constants $C > 0$ and $\delta > 0$ such that*

$$C^{-1} \left(\frac{|E|}{|I|} \right)^{p} \le \frac{\int_E u(x)\,dx}{\int_I u(x)\,dx} \le C \left(\frac{|E|}{|I|} \right)^{\delta}$$

for any I and for any measurable set $E \subset I$.

LEMMA 4.5. *If u belongs to A_p, $1 \le p < \infty$, then for $0 < \alpha \le 1$ there is a constant $C > 0$ such that*

$$\frac{1}{|I|} \int_I u(x)^\alpha\, dx \le \left(\frac{1}{|I|} \int_I u(x)\, dx \right)^{\alpha} \le C \frac{1}{|I|} \int_I u(x)^\alpha\, dx.$$

LEMMA 4.6. *If u belongs to A_p, $1 \le p < \infty$, then for $\beta > 0$ and for $0 < \gamma \le 1$ there is a constant $C > 0$ such that*

$$C^{-1} \le \frac{\int_r^{2r} \left(\int_{I(a,t)} u(x)\, dx \right)^{\gamma} \frac{dt}{t^{n\beta+1}}}{\int_{I(a,2r)\setminus I(a,r)} u(x)^{\gamma} |x-a|^{-n(1-\gamma+\beta)}\, dx} \le C,$$

for any $a \in \mathbb{R}^n$ *and* $r > 0$.

LEMMA 4.7. *If* u *belongs to* A_p, $1 \leq p < \infty$, *then for* $p' \geq p$ *and* $p' > 1$ *there is a constant* $C > 0$ *such that*

$$C^{-1} \leq \frac{\int_{I(a,R)\setminus I(a,r)} u(x)|x-a|^{-np'}\,dx}{r^{-np'}\int_{I(a,r)} u(x)\,dx} \leq C,$$

for any $a \in \mathbb{R}^n$ *and* $0 < 2r \leq R$.

PROOF OF THEOREM 4.2. By Lemma 4.4, w satisfies from (2.2.1) to (2.2.4). It follows from Lemma 4.6 that

$$\int_r^R \frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt$$

is comparable to

$$\int_{I(a,R)\setminus I(a,r)} u(x)^{\alpha/p}|x-a|^{-n(1-\alpha/p+1/p)}\,dx,$$

for $0 < 2r \leq R$. Therefore, we have (4.2.1) and (4.2.2).

PROOF OF THEOREM 4.3. If u belongs to A_q, then $u^{\alpha/p}$ belongs to $A_{(q-1)\alpha/p+1}$. Therefore, by Lemmas 4.6, 4.7 and 4.5, the following are comparable

$$\int_r^R \frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt,$$

$$\int_{I(a,R)\setminus I(a,r)} u(x)^{\alpha/p}|x-a|^{-n(1-\alpha/p+1/p)}\,dx,$$

$$r^{-n(1-\alpha/p+1/p)}\int_{I(a,r)} u(x)^{\alpha/p}\,dx,$$

$$r^{-n/p}w(a,r)^{1/p},$$

for $0 < 2r \leq R$. This shows (4.2.3).

V. Pointwise multipliers on the Lorentz spaces.

Let (X, μ) be a measure spase, and let A and B be linear spaces of functions defined of X. We denote the set of all pointwise multipliers from A to B by $PWM(A, B)$.

It is well known that, for $1/p = 1/p_1 + 1/p_2$, $1 \leq p \leq \infty$,

$$PWM(L^{p_1}(\mathbb{R}^n), L^p(\mathbb{R}^n)) = L^{p_2}(\mathbb{R}^n).$$

In this chapter, we generalize this equality to the Lorentz spaces.

1. Definitions.

Let (X, μ) be a σ-finite measure spase, and let $A, B \subset Meas(X)$ be complete quasi-norned linear spases. Assume that A and B have the generalized Cheby-shev's inequality (1.1.4), respectively. Then, in a way similar to Lemmas 1.3 and 1.4 in Chapter I, we can show that any pointwise multiplier from A to B is a bounded operator.

We recall the definitions of the Lorentz spaces. For a muasurable function f, we define the distribution function $\mu(f, s)$ as follows:

$$\mu(f, s) = \mu(\{x \in X : |f(x)| > s\}) \quad \text{for } s > 0.$$

And we denote by f^* the rearrangement of f:

$$f^*(t) = \inf\{s > 0 : \mu(f, s) \leq t\} \quad \text{for } t > 0.$$

This is a non-negative and non-increasing function which is continuous on the right and has the property

$$\mu(f^*, s) = \mu(f, s) \quad \text{for } s > 0.$$

If $|f(x)| \leq |g(x)|$ a.e. then

$$f^*(t) \leq g^*(t) \quad \text{for } t > 0.$$

Now the Lorentz space $L^{(p,q)}(X)$ is defined as follows: For $0 < p \leq \infty$ and $0 < q < \infty$,

$$L^{(p,q)}(X) = \left\{ f \in Meas(X) : \int_0^\infty t^{(q/p)-1}(f^*(t))^q \, dt < \infty \right\},$$

$$\|f\|_{L^{(p,q)}} = \left(\int_0^\infty t^{(q/p)-1}(f^*(t))^q \, dt \right)^{1/q}.$$

And for $0 < p \leq \infty$ and $q = \infty$,

$$L^{(p,\infty)}(X) = \left\{ f \in Meas(X) : \sup_{t>0} t^{1/p} f^*(t) < \infty \right\},$$

$$\|f\|_{L^{(p,\infty)}} = \sup_{t>0} t^{1/p} f^*(t).$$

Then, for $0 < p \leq \infty$, we have

$$L^{(p,p)}(X) = L^p(X) \quad \text{and} \quad \|f\|_{L^{(p,p)}} = \|f\|_{L^p}.$$

In general, $\|f\|_{L^{(p,q)}}$ is a quasi-norm and $L^{(p,q)}(X)$ is a complete quasi-normed linear space. If $1 < p < \infty$ and $1 \leq q \leq \infty$, then it is possible to replace the quasi-norm with a norm, which makes $L^{(p,q)}(X)$ a Banach space.

Moreover, the set of all simple functions is dense in $L^{(p,q)}(X)$ for $0 < p, q < \infty$.

Let $0 < p < \infty$ and $0 < q_1 \leq q_2 \leq \infty$. Then

$$L^{(p,q_1)}(X) \subset L^{(p,q_2)}(X)$$

and

$$\left(\frac{q_2}{p}\right)^{1/q_2} \|f\|_{L^{(p,q_2)}} \leq \left(\frac{q_1}{p}\right)^{1/q_1} \|f\|_{L^{(p,q_1)}}.$$

The Lorentz spaces have the generalized Chebyshev's inequality (1.1.4). So any pointwise multiplier from $L^{(p,q)}(X)$ to $L^{(p',q')}(X)$ is a bounded operator. However, since the Lorentz spaces have the property (1.1.2), we can use a method similar to Theorem 1.1.

2. Theorem.

Assume that X is expressible as a countable union of sets $X_i \subset X$ such that $\mu(X_i) < \infty$ $(i = 1,2,\ldots)$. Any bounded function whose support is included in X_i for some i is in $L^{(p,q)}(X)$. And $L^{(p,q)}(X)$ has the property (1.1.2). By Theorem 1.1 in Chapter **I**, we have

$$PWM(L^{(p,q)}(X)) = L^\infty(X) \quad \text{and} \quad \|g\|_{Op} = \|g\|_{L^\infty}.$$

This is a generalization of the equality

$$PWM(L^p(X)) = L^\infty(X).$$

It is well known that, for $1/p = 1/p_1 + 1/p_2$, $1 \le p \le \infty$,

$$PWM(L^{p_1}(\mathbb{R}^n), L^p(\mathbb{R}^n)) = L^{p_2}(\mathbb{R}^n).$$

We generalize this equality to the Lorentz spaces.

THEOREM 5.1. *Let* $0 < q \le p < \infty$ *and*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad q_1 = \frac{q}{p}p_1, \quad q_2 = \frac{q}{p}p_2.$$

Then

$$PWM(L^{(p_1,q_1)}(X), L^{(p,q)}(X)) = L^{(p_2,q_2)}(X) \quad \text{and} \quad \|g\|_{Op} = \|g\|_{L^{(p_2,q_2)}}$$

where $\|g\|_{Op}$ *is the operator norm of* $g \in PWM(L^{(p_1,q_1)}(X), L^{(p,q)}(X))$.

For $p = p_1 = p_2 = \infty$, let $q = q_1 = q_2 = \infty$. Then the above equality is trivial. If $q < \infty$ then $L^{(\infty,q)}(X) = \{0\}$. To prove this theorem, we show Lemmas.

LEMMA 5.2. *If* $\alpha \le 0$ *then*

$$\int_0^s t^\alpha (fg)^*(t)\,dt \le \int_0^s t^\alpha f^*(t) g^*(t)\,dt \quad \text{for } s > 0.$$

PROOF. Since $|f|^* = f^*$, we may assume that $f, g \ge 0$. Case 1: f and g are simple functions. Then $f^*g^* - (fg)^*$ is a step function. Since

$$\int_0^s f^*(t)g^*(t) - (fg)^*(t)\,dt \ge 0 \quad \text{for any } s > 0$$

(see for example [9, p.257]) and t^α is non-increasing, we have

$$\int_0^s t^\alpha (f^*(t)g^*(t) - (fg)^*(t))\,dt \ge 0 \quad \text{for any } s > 0.$$

Case 2: For any f and g, there are sequences $\{f_j\}$ and $\{g_j\}$ of simple functions such that

$$f_1 \leq f_2 \leq \ldots \quad \text{and} \quad f_j \to f \text{ a.e.,}$$
$$g_1 \leq g_2 \leq \ldots \quad \text{and} \quad g_j \to g \text{ a.e.}$$

Then

$$\int_0^s t^\alpha (f_j g_j)^*(t)\, dt \leq \int_0^s t^\alpha (f_j)^*(t)(g_j)^*(t)\, dt \leq \int_0^s t^\alpha f^*(t) g^*(t)\, dt.$$

Hence we have

$$\int_0^s t^\alpha (fg)^*(t)\, dt \leq \int_0^s t^\alpha f^*(t) g^*(t)\, dt.$$

LEMMA 5.3. *Let* $0 < q \leq p$ *and*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \qquad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

If $f \in L^{(p_1,q_1)}(X)$ *and* $g \in L^{(p_2,q_2)}(X)$ *then* $fg \in L^{(p,q)}(X)$ *and*

$$\|fg\|_{L(p,q)} \leq \|f\|_{L(p_1,q_1)} \|g\|_{L(p_2,q_2)}.$$

PROOF. If $q = 1$ then $p \geq 1$ and $(1/p) - 1 \leq 0$. By Lemma 5.2, we have

$$\begin{aligned}
&\int_0^\infty t^{(1/p)-1} (fg)^*(t)\, dt \\
&\quad \leq \int_0^\infty t^{(1/p)-1} f^*(t) g^*(t)\, dt = \int_0^\infty \left| t^{1/p_1} f^*(t) \right| \left| t^{1/p_2} g^*(t) \right| \frac{dt}{t} \\
&\quad \leq \left(\int_0^\infty \left(t^{1/p_1} f^*(t) \right)^{q_1} \frac{dt}{t} \right)^{1/q_1} \left(\int_0^\infty \left| t^{1/p_2} g^*(t) \right|^{q_2} \frac{dt}{t} \right)^{1/q_2}.
\end{aligned}$$

For $0 < q \leq p$, since $p/q \geq 1$, we have

$$\begin{aligned}
\|fg\|^q_{L(p,q)} &= \| |fg|^q \|_{L(p/q,1)} \\
&\leq \| |f|^q \|_{L(p_1/q,q_1/q)} \| |g|^q \|_{L(p_2/q,q_2/q)} = \|f\|^q_{L(p_1,q_1)} \|g\|^q_{L(p_2,q_2)}.
\end{aligned}$$

LEMMA 5.4. *Let* $0 < p, q < \infty$ *and*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad q_1 = \frac{q}{p} p_1, \quad q_2 = \frac{q}{p} p_2.$$

For any $g \in L^{(p_2,q_2)}(X)$, *there is* $f \in L^{(p_1,q_1)}(X)$ *such that*

$$\|fg\|_{L(p,q)} = \|f\|_{L(p_1,q_1)} \|g\|_{L(p_2,q_2)}.$$

PROOF. Let $f = g^{q_2/q_1}$. Then we have the above equality.

Now we prove Theorem 5.1.

PROOF OF THEOREM 5.1. If g is in $L^{(p_2,q_2)}(X)$ then, by Lemma 5.3, g is in $PWM(L^{(p_1,q_1)}(X), L(p,q)(X))$. Moreover, we have by Lemma 5.4

$$\|g\|_{Op} = \|g\|_{L(p_2,q_2)}.$$

Conversely, assume that g is in $PWM(L^{(p_1,q_1)}(X), L(p,q)(X))$. Let

$$g_i(x) = \begin{cases} 0 & x \in X \setminus X_i, \\ g(x) & x \in X_i \text{ and } |g(x)| \le i, \\ i & x \in X_i \text{ and } |g(x)| > i, \end{cases}$$

for $i = 1, 2, \dots$. Then g_i is in $L^{(p_2,q_2)}(X)$. By the first half of the proof, we have

$$\|g_i\|_{Op} = \|g_i\|_{L(p_2,q_2)}.$$

For any $f \in L^{(p_1,q_1)}(X)$, fg is in $L(p,q)(X)$ and $(fg_i)^* \le (fg)^*$. Hence

$$\|fg_i\|_{L(p,q)} \le \|fg\|_{L(p,q)} < \infty \quad \text{for } i = 1, 2, \dots.$$

By the uniform boundedness theorem, there is a constant M, $0 < M < \infty$, such that

$$\sup_i \|g_i\|_{L(p_2,q_2)} = \sup_i \|g_i\|_{Op} = M.$$

Since

$$g_1^* \le g_2^* \le \dots \quad \text{and} \quad g_i^* \to g^*$$

we have $g \in L^{(p_2,q_2)}(X)$.

The proof is complete.

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