STABLE DOUBLE POINT NUMBERS OF PAIRS OF SPHERICAL CURVES

TSUYOSHI KOBAYASHI AND SUMIKA KOBAYASHI

Abstract. For a pair of spherical curves P and P', we introduce a quantity called the stable double point number of P and P', denoted $sd(P, P')$. A trivial spherical curve is a spherical curve with no double point, denoted $\bigcirc$. In this paper, we prove two results relevant to the question whether $sd(P, \bigcirc) \leq d([P]) + 1$ holds or not, where $d([P])$ denotes the double point number. One is a positive result. Concreatly speaking, we show that each 2-bridge spherical curve denoted $C(a_1, a_2, \ldots, a_n)$ satisfies $sd(C(a_1, a_2, \ldots, a_n), \bigcirc) \leq d([C(a_1, a_2, \ldots, a_n)]) + 1$. The other is a negative result. Concreatly speaking, we show that the pretzel spherical curve denoted $P(m; 5)$ (m: odd integer, ≥ 5) satisfies $sd(P(m; 5), \bigcirc) = d([P(m; 5)]) + 2$.

1. Introduction

A *spherical curve* is the image of a generic immersion of a circle into a 2-sphere. A *trivial spherical curve* is a spherical curve with no double point. We say that a spherical curve P' is obtained from a spherical curve P by a deformation of type RI (type RII, type RIII resp.) if P' is obtained from P by replacing the part contained in a disk as in Figure 1. It is known that any two spherical curves can be transformed each other by a finite sequence of deformations of type RI, type RII, or type RIII, and ambient isotopies, i.e. for any pair of spherical curves P and P', there is a sequence of spherical curves

$$P = P_0 \to P_1 \to \cdots \to P_n = P'$$

such that P_{i+1} is obtained from P_i ($i = 0, 1, \ldots, n-1$) by a deformation of type RI, RII or RIII (up to ambient isotopy).

Suppose that a spherical curve P' is obtained from a spherical curve P by a deformation of type RI. We say that P' is obtained from P by a deformation of type

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RI$^+$ (type RI$^-$ resp.), if the number of double points of P' is greater (less resp.) than that of P. See Figure 1.

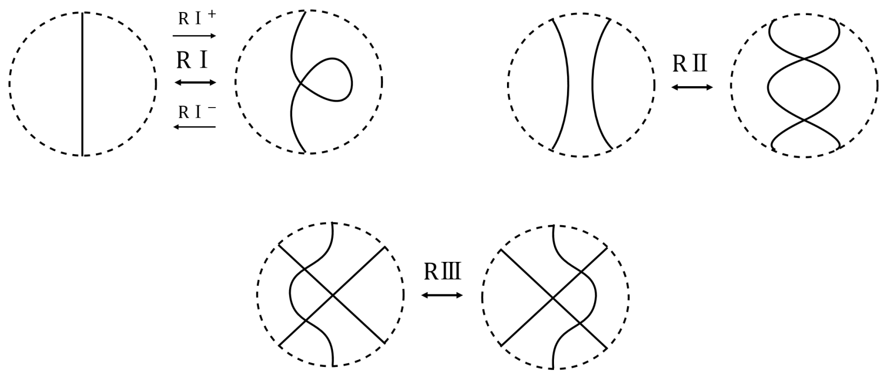


Figure 1. Deformations of type RI, RII, RIII

In 2001, Östlund [7] had asked whether, for any plane curve P, a trivial plane curve is obtained from P by a sequence of deformations of type RI or type RIII, or not. Then Hagge and Yazinski [3] gave a negative answer to this question. In fact, they showed that the plane curve P_{HY} in Figure 2 can not be transformed to a trivial plane curve by any sequence of deformations of type RI and type RIII. Although the original question is concerned with plane curves, it is natural to ask the same issue for spherical curves. This was studied by Ito and Takimura [5] [6], and they showed the following.

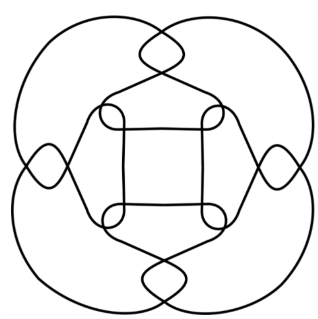


Figure 2. P_{HY}

We say that spherical curves P and P' are *RI, RIII-equivalent* (or P' is *RI, RIII-equivalent to* P) if there is a sequence of spherical curves

$$P = P_0 \to P_1 \to \cdots \to P_n = P'$$

such that P_{i+1} is obtained from P_i ($i = 0, 1, \ldots, n-1$) by a deformation of type RI, or RIII (up to ambient isotopy). In [5], Ito and Takimura showed that there are infinitely many spherical curves that are not RI, RIII-equivalent to a trivial spherical curve. Further in [6], they showed that there are infinitely many spherical curves that are not mutually RI, RIII-equivalent. This result leads us to:

Problem. Study the pairs of spherical curves that are RI, RIII-equivalent, or not RI, RIII-equivalent.

In this paper, we introduce a 1-complex for studying the problem by mimicking the arguments in [1]. Let us introduce the construction. Let $\mathcal{C}$ be the set of the ambient isotopy classes of the spherical curves. We say that two elements v and v' of $\mathcal{C}$ are *RI-equivalent*, denoted by $v \sim_{RI} v'$, if there are representatives P, P' of v, v' respectively such that P' is obtained from P by a sequence of deformations of type RI and ambient isotopies. We note that $\sim_{RI}$ is an equivalence relation on $\mathcal{C}$ (see Proposition 1.1 of [1]). Then $\widetilde{\mathcal{C}}$ denotes the quotient set $\mathcal{C}/\sim_{RI}$ and for a spherical curve P, $[P]$ denotes the equivalence class containing the ambient isotopy class of P. Then we obtain a 1-complex, denoted $\widetilde{\mathcal{K}}_3$ by:

- the set of vertices of $\widetilde{\mathcal{K}}_3$ corresponds to $\widetilde{\mathcal{C}}$, and
- two vertices v and v' are joined by an edge if there are representatives P and P' of v and v' respectively such that P' is obtained from P by a sequence consisting of exactly one deformation of type RIII, and some (possibly, empty) deformation(s) of type RI (up to ambient isotopy).

We note that $\widetilde{\mathcal{K}}_3$ is not connected. In fact, the above mentioned result in [6] implies that there are infinitely many components in $\widetilde{\mathcal{K}}_3$. In this paper, we propose more subtle treatment of $\widetilde{\mathcal{K}}_3$. Concreatly speaking, for a pair of spherical curves (P, P'), we introduce the quantity $sd(P, P')$ called stable double point number of P and P' as follows.

Let $\widetilde{\mathcal{C}}$, $\widetilde{\mathcal{K}}_3$ be as above. For each $v \in \widetilde{\mathcal{C}}$ (: the vertices of $\widetilde{\mathcal{K}}_3$), we define the *double point number* of v, denoted $d(v)$, by

$$d(v) := \min\{\text{the number of the double points of } P | P : \text{ a representative of } v\}.$$

For a pair of spherical curves $(P,\ P')$, we define the *stable double point number* of $(P,\ P')$, denoted $sd(P, P')$, as follows.

Let $\mathcal{L}(P, P')$ be the set of paths in $\widetilde{\mathcal{K}}_3$ connecting $[P]$ and $[P']$. For $L \in \mathcal{L}(\mathcal{P}, \mathcal{P}')$, $V(L)$ denotes the set of the vertices of L. Then, we define $sd(P, P')$ by:

$$sd(P, P') := \min_{L \in \mathcal{L}(P, P')} \{ \max_{v \in V(L)} \{d(v)\}\}$$

if P and P' are RI, RIII-equivalent, and $sd(P, P') := \infty$ if P and P' are not RI, RIII-equivalent.

Remark 1.1. It is clear from the definition, that

$$sd(P, P') \geq \max\{d([P]), d([P'])\}.$$

Let P be a spherical curve. Each component of $S^2 \setminus P$ is called a *region* of P. Let R be a region of P. We call R an *n-gon*, if R has n corners. In particular, 3-gon is called a triangle. The next proposition shows that there exist many pairs of spherical curves (P, P') such that $sd(P, P') < \infty$, and $sd(P, P') > \max\{d([P]), d([P'])\}$.

Proposition 1.1. *Let $(P,\ P')$ be a pair of spherical curves such that $[P] \neq [P']$, $d([P']) \leq d([P])$. Suppose that each region of P is not a 1-gon or a triangle. Then we have:*

$$sd(P, P') \geq d([P]) + 1.$$

By Proposition 1.1, it is natural to ask:

Question. For any pair of spherical curves (P, P'), does the inequality

$$sd(P, P') \leq \max\{d([P]), d([P'])\} + 1$$

hold? In particular, for each spherical curve P that is RI, RIII-equivalent to $\bigcirc$, does the inequality

$$sd(P, \bigcirc) \leq d([P]) + 1$$

hold, where $\bigcirc$ denotes a trivial spherical curve?

We give two results concerning the question. One is a positive result. In Theorem 3.1, we show that each 2-bridge spherical curve denoted $C(a_1, a_2, \ldots, a_n)$ (for the definition, see Section 3.2) satisfies:

$$sd(C(a_1, a_2, \ldots, a_n), \bigcirc) = \begin{array}{l} d([C(a_1, a_2, \ldots, a_n)]), \text{ or} \\ d([C(a_1, a_2, \ldots, a_n)]) + 1. \end{array}$$

The other is a negative result. In Theorem 3.2, we show that the pretzel spherical curve (for the definition, see Section 3.4) denoted $P(m; 5)$ (m: odd integer, ≥ 5) satisfies:

$$sd(P(m; 5), \bigcirc) = d([P(m; 5)]) + 2.$$

2. Preliminaries

2.1. **Deformation of type α and type β.** The spherical curve P is called *RI-minimal* if each region of P is not a 1-gon. For any spherical curve P, we obtain an RI-minimal spherical curve by successively applying deformations of type RI$^-$. It is known that such spherical curves are mutually ambient isotopic, and reduced(P) denotes such a spherical curve.

Remark 2.1. Let P be a spherical curve. Recall from Section 1 that $d([P])$ denotes the double point number of the RI-equivalence class containing P. Then it is elementary to show:

$$d([P]) = \text{the number of the double points of reduced}(P).$$

Definition 2.1 (Deformations of type α)**.** For spherical curves P and P', we say that P' is obtained from P by a *deformation of type* α, if P' is obtained by replacing the part of P contained in a disk as in Figure 3. We say that P' is obtained from P by a deformation of type α^+ (type α^- resp.), if the number of double points of P' is greater (less resp.) than that of P. See Figure 3.

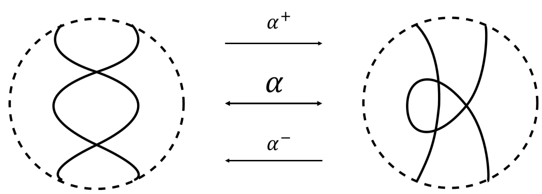


FIGURE 3. Deformation of type α

Remark 2.2. We note that the deformation of type α is realized by using a deformation of type RI and a deformation of type RIII. See Figure 11 of [1].

Further in [4], another deformation called a *deformation of type* β (type $\beta^{\pm}$ resp.) is defined. Roughly speaking, we say that P' is obtained from P by a deformation of type β^+, if P' is obtained from P by successively applying connected sum with (possibly, empty) spherical curve(s) with one double point(s) and one spherical curve of trefoil type in a non trivial manner. See Figure 4. For the precise definition, see [4]. Note that P' contains linearly arrayed 3 regions consisting of 2-gon, triangle, and 2-gon corresponding to shaded regions in Figure 4. We call the union of such regions a *β-realm*. Then it clear from Figure 4, we cannot perform a deformation of type β^- unless there exists β-realm. Then the next proposition is proved in [4].

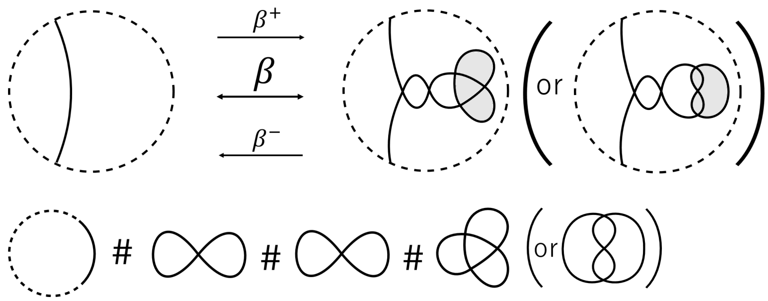


FIGURE 4. Deformation of type β

Proposition 2.1 ([4])**.** *Let P, P' be spherical curves. Then P' is obtained from P by a sequence consisting of one deformation of type RIII, and some deformation(s) of type RI (i.e., $[P]$ and $[P']$ are adjacent in $\widetilde{\mathcal{K}}_3$) if and only if reduced(P') is obtained from reduced(P) by exactly one deformation that is of type RIII, type α, or type β (up to ambient isotopy).*

2.2. **Specifications for the deformations.** Let $P(\subset S^2)$ be a spherical curve. Suppose that P' is obtained from P by a deformation of type RI$^+$. Then we may suppose, by an ambient isotopy, that the deformation is performed in a very small region, and we will specify the place where the deformation is performed by a dot on P. See Figure 5. We call it *the dot relevant to the deformation* (of type RI$^+$).

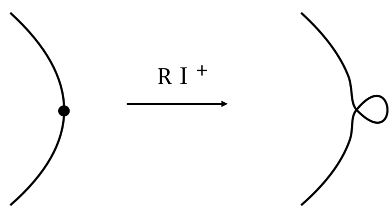


FIGURE 5. The dot relevant to the deformation of type RI$^+$

Suppose that P' is obtained from P by a deformation of type RI$^-$. Then we can specify the 1-gon contained in the disk where the deformation is performed. See Figure 6. We call it *the 1-gon relevant to the deformation* (of type RI$^-$).

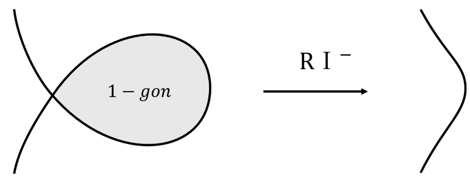


FIGURE 6. The 1-gon relevant to the deformation of type RI$^-$

Suppose that P' is obtained from P by a deformation of type RⅢ. Then we can specify the triangle contained in the disk where the deformation is performed. See Figure 7. We call it *the triangle relevant to the deformation* (of type RⅢ).

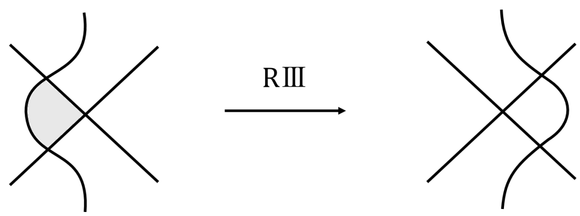


FIGURE 7. The triangle relevant to the deformation of type RⅢ

Suppose that P' is obtained from P by a deformation of type α^-. Then we can specify the union of a 2-gon and a triangle contained in the disk where the deformation is performed. See Figure 8. We call the union of the 2-gon and the triangle *the 1-gon with a crack relevant to the deformation* (of type α^-).

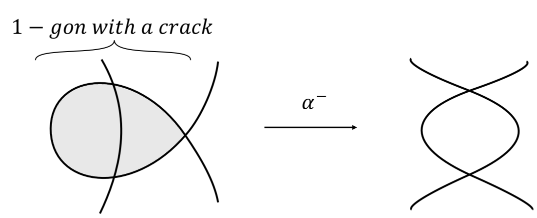


FIGURE 8. The 1-gon with a crack relevant to the deformation of type α^-

Suppose that P' is obtained from P by a deformation of type α^+. Then we can specify the 2-gon contained in the disk where the deformation is performed. Here we note that there are two essentially different ways of performing α^+ along the 2-gon (Figure 9).

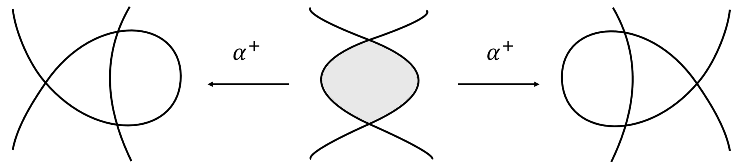


FIGURE 9. Two deformations of type α^+, which are essentially defferent

We will specify the type of the deformation by adding a dotted arc that is parallel to an edge of the 2-gon as in Figure 10. We call the union of them *the 2-gon with a dotted arc relevant to the deformation* (of type α^+). We may regard that P' is obtained from P by squeezing the endpoints of the dotted arc along it. See Figure 11.

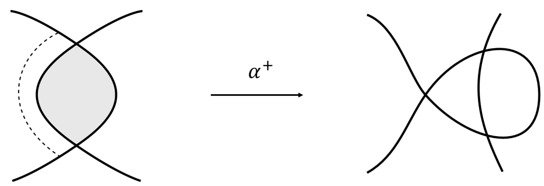


FIGURE 10. The 2-gon with a dotted arc relevant to the deformation of type α^+

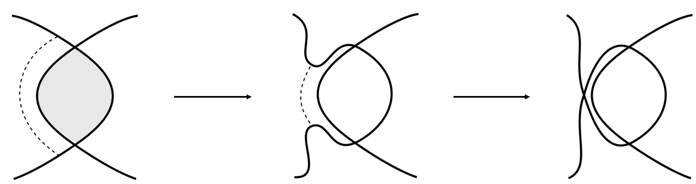


FIGURE 11. Squeezing the endpoints of the dotted arc

Let E be a subsurface in S^2 such that ∂E is in a general position with respect to P. Suppose that P' is obtained from P by a deformation of type RI$^+$ (RI$^-$, RIII, α^+, α^- resp.). We say that *the deformation is performed within* E if the dot (1-gon, triangle, 1-gon with a crack, 2-gon with a dotted arc resp.) relevant to the deformation is contained in E. The next proposition may look clear to the reader, but we think it is worth giving the statement to understand the proof of Theorem 3.2.

Proposition 2.2. *Let P, P', E be as above. Suppose that the deformation is performed within E. Then there exists a disk D in E such that there is a spherical curve $\widetilde{P'}$ satisfying the following conditions.*

(1) *$\widetilde{P}'$ is ambient isotopic to P',*
(2) *$\widetilde{P}' \setminus D = P \setminus D$, and*
(3) *$\widetilde{P}'$ is obtained from P by replacing the part $P \cap D$ as in Figure 1 or Figure 3.*

Proof. We divide the proof into the following cases.

Case 1: The deformation is of type RI^+.

Since the dot relevant to the deformation is contained in E, the proposition is clear.

Case 2: The deformation is of type RI^-.

We may regard that the deformation is performed in a very small disk containing the double point adjacent to the 1-gon, and this gives the conclusion of the proposition. See Figure 12.

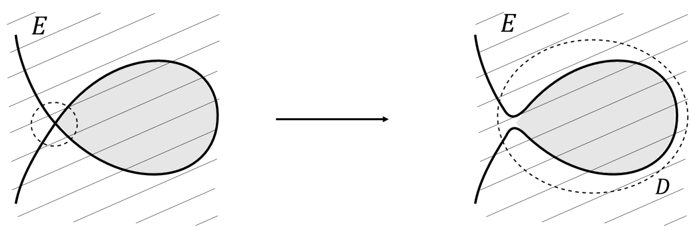


FIGURE 12. The deformation of type RI^- is realized within E

Case 3: The deformation is of type RⅢ.

By Figure 13, we see that the proposition holds.

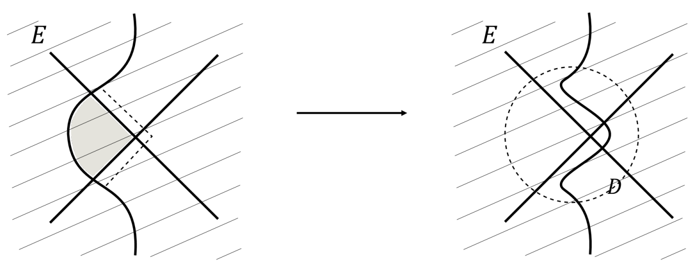


FIGURE 13. The deformation of type RⅢ is realized within E

Case 4: The deformation is of type α^-.

In this case, we may regard that $\widetilde{P}'$ is obtained from P by deforming P in a small disk containing the double point adjacent to the 1-gon with a crack as in Figure 14, and this shows that the proposition holds.

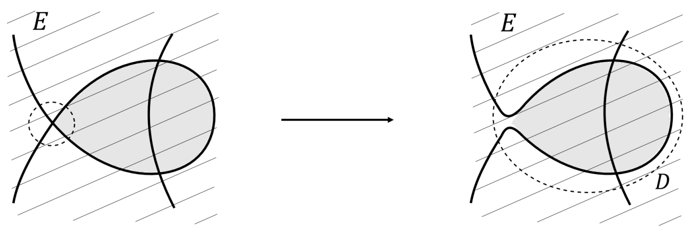


FIGURE 14. The deformation of type α^- is realized within E

Case 5: The deformation is of type α^+.

In this case, since we may suppose the dotted arc relevant to the deformation is contained in E, we see that the proposition holds. See Figure 15.

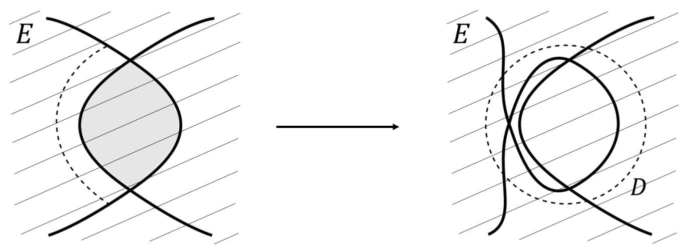


FIGURE 15. The deformation of type α^+ is realized within E

□

3. RESULTS

In this section, we give proofs of the results of this paper (Proposition 1.1, Theorems 3.1, 3.2).

3.1. Proof of Proposition 1.1.

Proof. Let $P = P_0 \to P_1 \to \cdots \to P_n = P'$ be a sequence of spherical curves such that P_{i+1} is obtained from P_i by a deformation of type RI or type RIII ($i = 0, 1, \ldots, n-1$). Suppose that $P_j \to P_{j+1}$ is the first deformation of type RIII in the sequence (hence, the deformations in $P_0 \to P_1 \to \cdots \to P_j$ are of type RI). Hence [reduced(P_j)]=[reduced(P)]. The assumption of the proposition (each region of P is not a 1-gon) implies that reduced(P)=P. Then by Proposition 2.1, we may suppose that reduced(P_{j+1}) is obtained from reduced(P_j) by a deformation of type RIII, type α, or type β. These facts show that

$(*)$ reduced(P_{j+1}) is obtained from P by a deformation of type RIII , type α, or type β.

Then we note that each region of P is not a triangle by the assumption of the proposition. This shows that the deformation $P \to$ reduced(P_{j+1}) is not of type RIII, type α^-, or type β^-. If the deformation is of type α^+, then by Remark 2.1 and $(*)$ we have

$$d([P_{j+1}]) = d([P]) + 1.$$

If the deformation is of type β^+, then by Remark 2.1 and $(*)$ we have

$$d([P_{j+1}]) \geq d([P]) + 3.$$

These imply that $sd(P, P') \geq d([P]) + 1$, and this completes the proof of the proposition. $\square$

3.2. **2-bridge spherical curves.** For an n-tuple of positive integers $a_1, a_2, \ldots, a_n$ $(n \geq 1)$, let $C(a_1, a_2, \ldots, a_n)$ be a curve as in Figure 16, where $C(a_1, a_2, \ldots, a_n)$ intersects each rectangular region B_i $(i = 1, 2, \ldots, n)$ in curves as in Figure 16. It is known that if $(a_1, a_2, \ldots, a_n)$ satisfies the following condition, then $C(a_1, a_2, \ldots, a_n)$ is a spherical curve. (See Proposition 1 of [2].)

Consider the irreducible fraction p/q derived from $(a_1, a_2, \ldots, a_n)$ by using the continued fraction

$$\frac{p}{q} = a_1 + \cfrac{1}{a_2 + \frac{1}{\cdots + \frac{1}{a_n}}}.$$

Then p is an odd integer.

We call such spherical curve $C(a_1, a_2, \ldots, a_n)$ a *2-bridge spherical curve* of type $(a_1, a_2, \ldots, a_n)$.

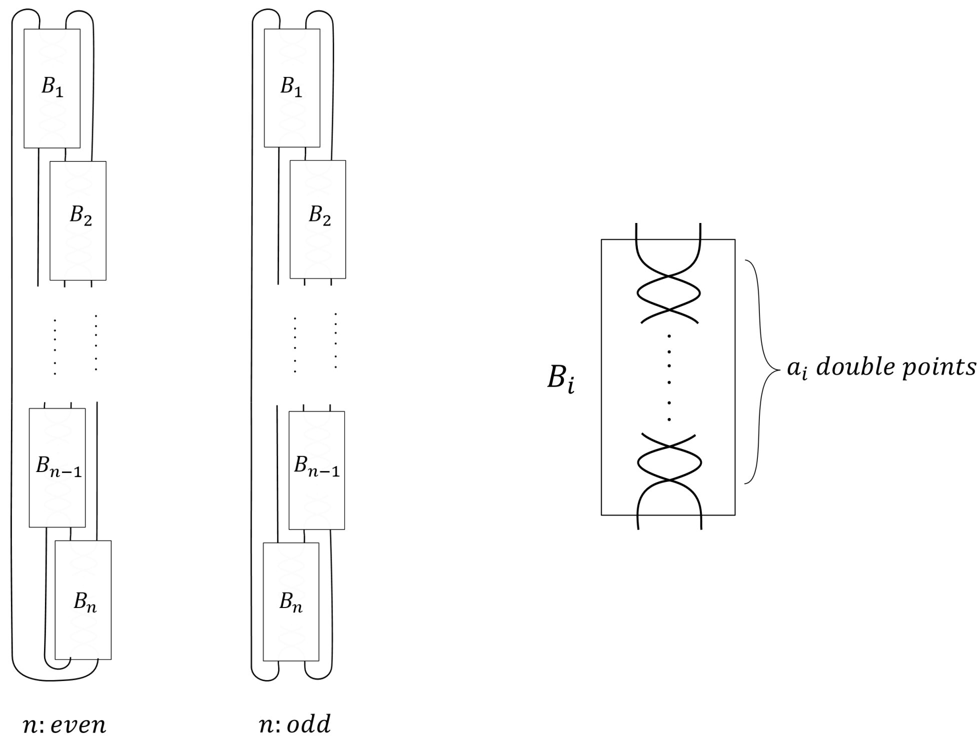


FIGURE 16. $C(a_1, a_2, \ldots, a_n)$

Remark 3.1. It is directly observed from Figure 16, that

$$\begin{aligned} C(1, a_2, a_3, \ldots, a_n) &= C(a_2 + 1, a_3, \ldots, a_n), \\ C(a_1, a_2, \ldots, a_{n-1}, 1) &= C(a_1, a_2, \ldots, a_{n-1} + 1). \end{aligned}$$

By Remark 3.1, we may suppose $a_1 > 1$ and $a_n > 1$, unless $n = 1$, $a_1 = 1$.

Remark 3.2. It is directly observed from Figure 16, that

$$d([C(a_1, a_2, \ldots, a_n)]) = a_1 + a_2 + \cdots + a_n$$

unless $n = 1$, $a_1 = 1$.

It is shown in Proposition 2 of [6] that each $C(a_1, a_2, \ldots, a_n)$ is RI, RIII-equivalent to a trivial spherical curve. We give a strengthened version of this result, which studies the stable double point numbers.

Theorem 3.1. *For each 2-bridge spherical curve $C(a_1, a_2, \ldots, a_n)$, we have*

$$\begin{aligned} sd(C(a_1, a_2, \ldots, a_n), \bigcirc) = \ & d([C(a_1, a_2, \ldots, a_n)])(= a_1 + a_2 + \cdots + a_n), \text{ or} \\ & d([C(a_1, a_2, \ldots, a_n)]) + 1(= a_1 + a_2 + \cdots + a_n + 1) \end{aligned}$$

unless $n = 1$, $a_1 = 1$.

Proof. We prove the theorem by the induction on n. We remark that we first prove the theorem for the case of $n = 2$. Then we prove for the case of $n = 1$, and general cases.

Suppose that $n = 2$. Since $C(a_1, a_2)$ is a spherical curve, we may suppose, without loss of generality, that a_1 is an even number. We devide the proof into the following two cases.

Case 1: $a_1 = 2$.

In this case, we see that there is a 1-gon with a crack as in Figure 17 (a). Hence we can apply the deformation of type α^- by using the 1-gon with a crack to obtain the 2-bridge spherical curve $C(2, a_2 - 1)$. By repeatedly applying such deformations a_2 times, we obtain a spherical curve as in Figure 17 (b). Then by applying deformations of type RI^- twice, we obtain a trivial spherical curve (Figure 17 (c)). It is easy to see that these deformations imply

$$sd(C(2, a_2), \bigcirc) = 2 + a_2.$$

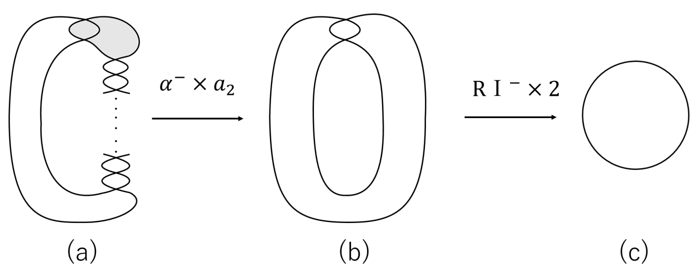


FIGURE 17. $C(2, a_2)$

Case 2: $a_1 > 2$.

In this case, we first apply the deformation of type α^+ by using the 2-gon with the dotted arc as in Figure 18 (a) to obtain the spherical curve of Figure 18 (b). Then we apply a deformation of type RIII by using the triangle in Figure 18 (b). Further we repeatedly apply deformations of type RIII $a_1 - 3$ times within B_1 to obtain the spherical curve of Figure 18 (c-1). It is directly observed that the spherical curve of Figure 18 (c-2) is ambient isotopic to that of Figure 18 (c-1). Then we apply a deformation of type RIII by using the triangle in Figure 18 (c-2). Further we repeatedly apply deformations of type RIII $a_2 - 1$ times within B_2 to obtain the spherical curve of Figure 18 (d). Then we apply the deformation of type α^- by using the 1-gon with a crack as in Figure 18 (d) to obtain the spherical curve as in Figure 18 (e), that is $C(a_1 - 2, a_2 + 2)$. These show that $sd(C(a_1, a_2), C(a_1 - 2, a_2 + 2)) \leq a_1 + a_2 + 1$. On the other hand, by Proposition 1.1, we see that $sd(C(a_1, a_2), C(a_1 - 2, a_2 + 2)) \geq a_1 + a_2 + 1$, and these show that $sd(C(a_1, a_2), C(a_1 - 2, a_2 + 2)) = a_1 + a_2 + 1$.

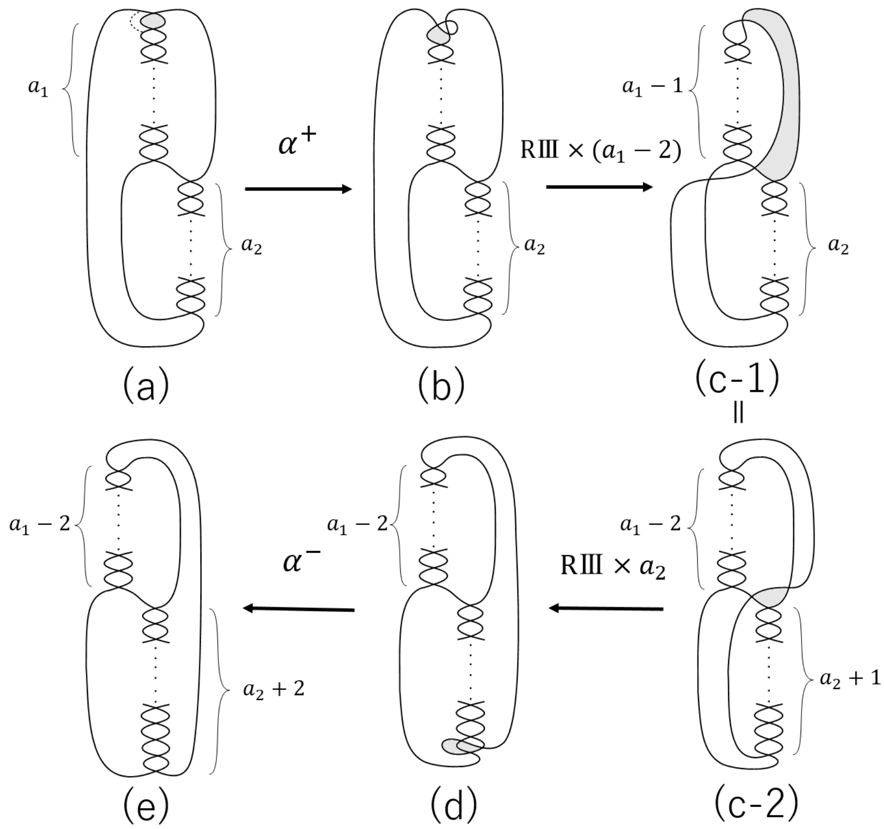


FIGURE 18. $C(a_1, a_2)(a_1 > 2)$

By repeatedly applying the deformations of Figure 18 $(a_1 - 2)/2$ times on $C(a_1, a_2)$, we obtain $C(2, a_1 + a_2 - 2)$. Further we see that $sd(C(a_1, a_2), C(2, a_1 + a_2 - 2)) = a_1 + a_2 + 1$. This together with Case 1 above and Proposition 1.1, we see that

$$sd(C(a_1, a_2), \bigcirc) = a_1 + a_2 + 1.$$

These complete the proof of Theorem 3.1 for the case of $n = 2$.

Suppose that $n = 1$. Since $C(a_1)$ is a spherical curve, we see that a_1 is an odd integer. If $a_1 = 3$, by Figure 19, we see that $sd(C(3), \bigcirc) = 3$.

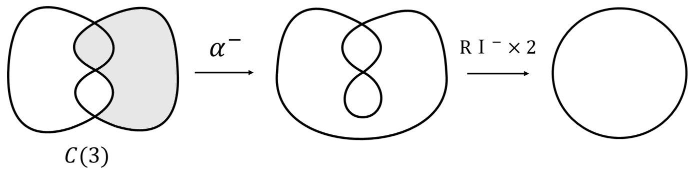


FIGURE 19. $C(3)$

Suppose that $a_1 > 3$. By Figure 20, we see that we can obtain $C(3, a_1 - 3)$ from $C(a_1)$ by successively applying α^+, RⅢ, and α^-, and this shows that $sd(C(a_1), C(3, a_1 - 3)) \leq a_1 + 1$. On the other hand, by Proposition 1.1, we see that $sd(C(a_1), C(3, a_1 - 3)) \geq a_1 + 1$, and these show that $sd(C(a_1), C(3, a_1 - 3)) = a_1 + 1$. Then by Case 2 above and Proposition 1.1, we see that

$$sd(C(a_1), \bigcirc) = a_1 + 1.$$

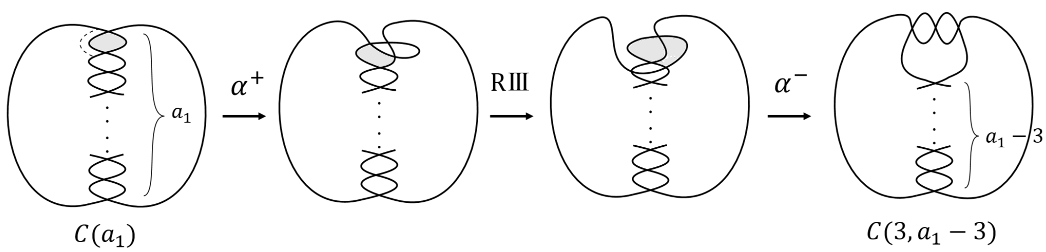


FIGURE 20. $C(a_1)(a_1 > 3)$

These complete the proof of Theorem 3.1 for the case of $n = 1$.

Now we consider the case of $n > 2$. We suppose that Theorem 3.1 holds for the case of $n \leq k$. Suppose that $n = k + 1$. We divide the proof into the following two cases.

Case A: $a_1 = 2$.
By Figure 21, we see that

$$\begin{aligned} sd(C(2, a_2, \ldots, a_n), C(a_3 + 2, a_4, \ldots, a_n)) &= 2 + a_2 + \cdots + a_n \\ &= a_1 + a_2 + \cdots + a_n. \end{aligned}$$

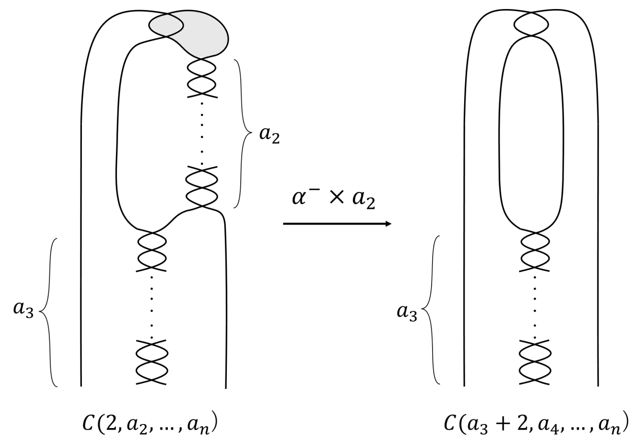


FIGURE 21. $C(2, a_2, \ldots a_n)$

Then the assumption of the induction implies

$$\begin{aligned} sd(C(a_3+2, a_4, \ldots, a_n), \bigcirc) &\leq (a_3+2) + a_4 + \cdots + a_n + 1 \\ &\leq 2 + a_2 + a_3 + \cdots + a_n \\ &= a_1 + a_2 + \cdots + a_n. \end{aligned}$$

These show that

$$\begin{aligned} sd(C(2, a_2, \ldots, a_n), \bigcirc) &= 2 + a_2 + \cdots + a_n \\ &= a_1 + a_2 + \cdots + a_n. \end{aligned}$$

Case B: $a_1 > 2$.
We first prove the next claim.

Claim 3.1.

$$sd(C(a_1, a_2, \ldots, a_{n-1}, a_n), C(a_1-2, a_2, \ldots, a_{n-1}, a_n+2)) \leq a_1+a_2+\cdots+a_{n-1}+a_n+1.$$

Proof of Claim 3.1. Let P_1 be the spherical curve obtained from $C(a_1, a_2, \ldots, a_n)$ by applying the deformations of Figure 22.

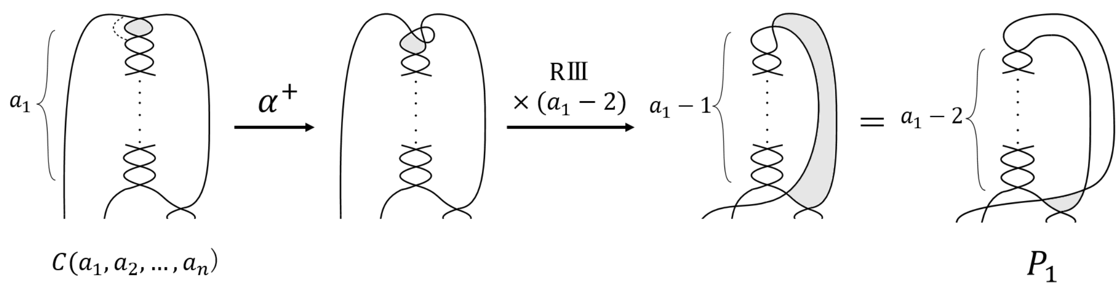


FIGURE 22. $C(a_1, a_2, \dots, a_n) \to P_1$

By Figure 22, we see that P_1 is transformed into the spherical curve P_2 in Figure 23 (b) by a sequence consisting of a_2 deformations of type RⅢ.

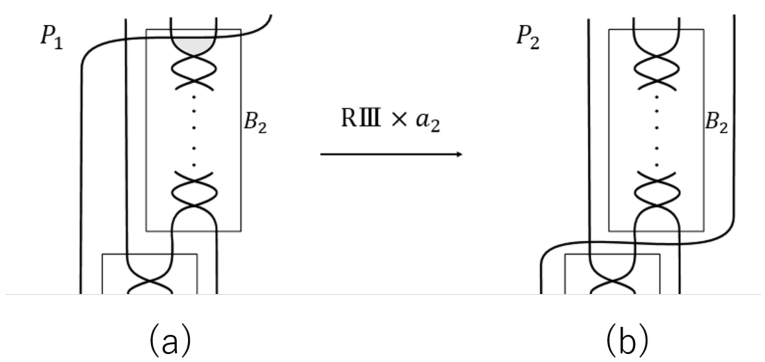


FIGURE 23. $P_1 \to P_2$

We further apply the deformations as in Figure 23 $(n-3)$ times to obtain the spherical curve P_{n-1} as in Figure 24, where B_1 contains $(a_1 - 2)$ double points of P_{n-1}, and B_j $(j = 2, 3, \dots, n-1)$ contains a_j double points of P_{n-1}.

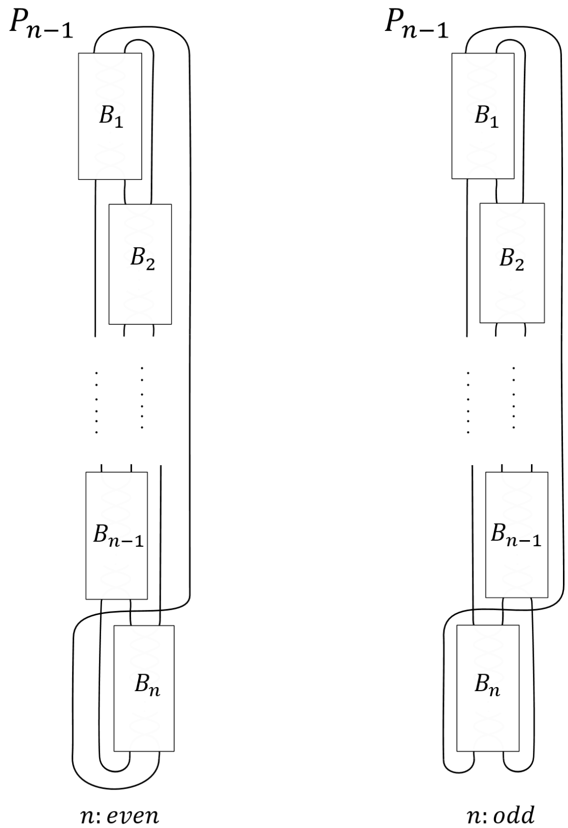


FIGURE 24. P_{n-1}

If n is an odd number, then we apply the deformations as in Figure 25 to obtain $C(a_1 - 2, a_2, \ldots, a_{n-1}, a_n + 2)$, and this shows that

$$sd(C(a_1, a_2, \ldots, a_{n-1}, a_n), C(a_1-2, a_2, \ldots, a_{n-1}, a_n+2)) \leq a_1+a_2+\cdots+a_{n-1}+a_n+1.$$

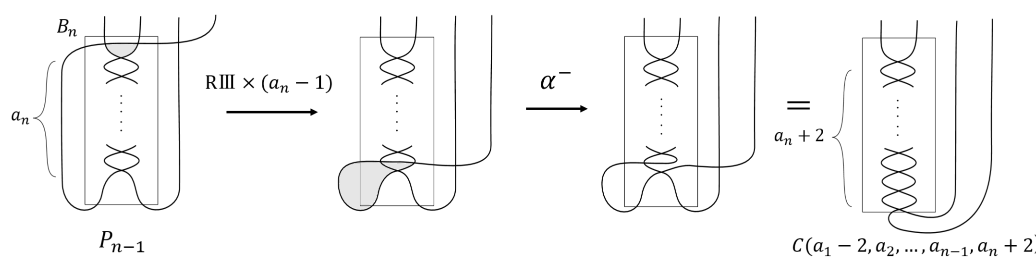


FIGURE 25. $P_{n-1} \to C(a_1 - 2, a_2, \ldots, a_{n-1}, a_n + 2)$($n$: odd)

If n is an even number, then we apply the deformations as in Figure 26 to obtain $C(a_1-2, a_2, \ldots, a_{n-1}, a_n+2)$, and this shows that

$$sd(C(a_1, a_2, \ldots, a_{n-1}, a_n), C(a_1-2, a_2, \ldots, a_{n-1}, a_n+2)) \leq a_1+a_2+\cdots+a_{n-1}+a_n+1.$$

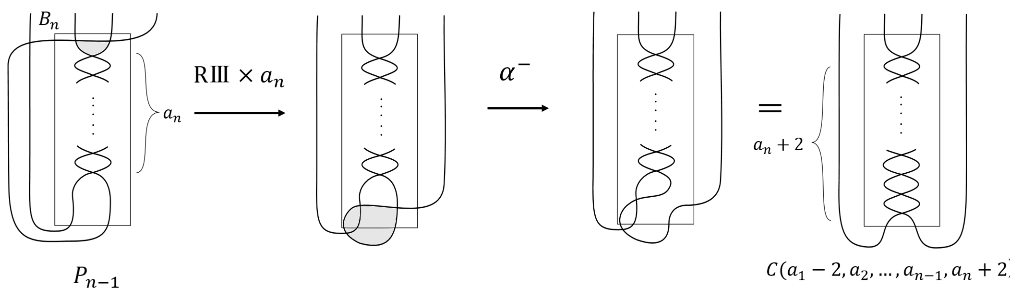


FIGURE 26. $P_{n-1} \to C(a_1-2, a_2, \ldots, a_{n-1}, a_n+2)$($n$: even)

These complete the proof of Claim 3.1. $\square$

If a_1 is an odd number, then we repeatedly apply the deformations of Claim 3.1 to $C(a_1, a_2, \ldots, a_{n-1}, a_n)$ $(a_1-1)/2$ times we see that

$$sd(C(a_1, a_2, \ldots, a_{n-1}, a_n), C(1, a_2, \ldots, a_{n-1}, a_n+a_1-1)) \leq a_1+a_2+\cdots+a_{n-1}+a_n+1.$$

Since $C(1, a_2, \ldots, a_{n-1}, a_n+a_1-1) = C(a_2+1, a_3, \ldots, a_{n-1}, a_n+a_1-1)$, we can apply the induction to show that

$$\begin{aligned} sd(C(1, a_2, \ldots, a_{n-1}, a_n+a_1-1), \bigcirc) &\leq 1+a_2+\cdots+a_{n-1}+(a_n+a_1-1)+1 \\ &= a_1+a_2+\cdots+a_{n-1}+a_n+1. \end{aligned}$$

These show that

$$sd(C(a_1, a_2, \ldots, a_{n-1}, a_n), \bigcirc) \leq a_1+a_2+\cdots+a_{n-1}+a_n+1.$$

If a_1 is an even number, then we repeatedly apply the deformations of Claim 3.1 to $C(a_1, a_2, \ldots, a_{n-1}, a_n)$ $(a_1-2)/2$ times we see that

$$sd(C(a_1, a_2, \ldots, a_{n-1}, a_n), C(2, a_2, \ldots, a_{n-1}, a_n+a_1-2)) \leq a_1+a_2+\cdots+a_{n-1}+a_n+1.$$

Then we apply the arguments in Case A above to show that

$$\begin{aligned} sd(C(2, a_2, \ldots, a_{n-1}, a_n+a_1-2), \bigcirc) &= 2+a_2+\cdots+a_{n-1}+(a_n+a_1-2) \\ &= a_1+a_2+\cdots+a_{n-1}+a_n. \end{aligned}$$

These show that

$$sd(C(a_1, a_2, \ldots, a_{n-1}, a_n), \bigcirc) \leq a_1+a_2+\cdots+a_{n-1}+a_n+1.$$

These complete the proof of Theorem 3.1. $\square$

3.3. **Deformation of type ξ_p.** For the statement of the second result, we introduce a new deformation of spherical curves. Let p be a positive odd integer. We say that a spherical curve P' is obtained from a spherical curve P by a *deformation of type* ξ_p if P' is obtained by replacing the part of P contained in a disk D as in Figure 27.

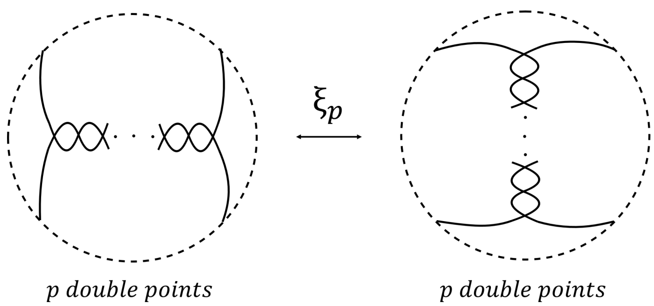


FIGURE 27. Deformation of type ξ_p

We note that the deformation of type ξ_p is exactly the deformation denoted $T(2k-1)$ with $2k-1=p$, in [6]. In Lemma 2 of [6], it is shown that the deformation of type ξ_p is realized by a sequence consisting of deformations of type RI or type RIII (up to ambient isotopy). In fact, Figure 29 depicts the sequence of deformations. Here we remark the next fact concerning the numbers of the double points of the spherical curves in the sequence. Let us consider the sequence of the spherical curves in Figure 29. Then we have:

Fact 3.1. The maximal number of double points of the spherical curves that are contained in the disk D is $p+(p-1)/2$.

Proof. We prove the fact by the induction on p. Suppose that $p=3$. In this case, the sequence is as in Figure 28. It is directly obseved from Figure 28 that the maximal number of the double points is $4(=3+(3-1)/2)$.

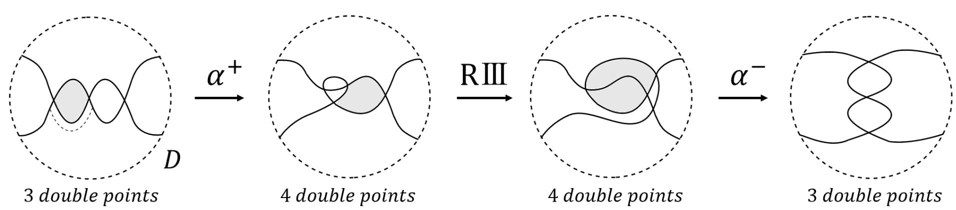


FIGURE 28. Deformation of type ξ_3

Suppose that the fact holds for $p=2l+1$. Then we consider the case of $p=2l+3$ (Figure 29). We first apply one deformation of type α^+, and $2l+1$ deformations of type RIII as in (a), (b), (c) of Figure 29 to obtain the spherical curve P_2 depicted in Figure 29 (c). Here we note that the number of the double points is raised by 1. Then we apply a deformation of type ξ_{2l+1} within the disk D' depicted in Figure 29 (c) to obtain the spherical curve P_3 in Figure 29 (d). Here we note that the number of the double points is raised by l within D', hence the number of the double points is raised by $l+1$ within D. Then we apply deformations of type RIII $2l$ times to obtain the spherical curve P_4 in Figure 29 (e). Then we apply the deformation of type α^- by using the 1-gon with a crack depicted in Figure 29 (e) to obtain the spherical curve P_5 (=P') in Figure 29 (f). These show that the maximal number of the double points contained in D is $(2l+3)+(l+1)(=(2l+3)+\{(2l+3)-1\}/2)$.

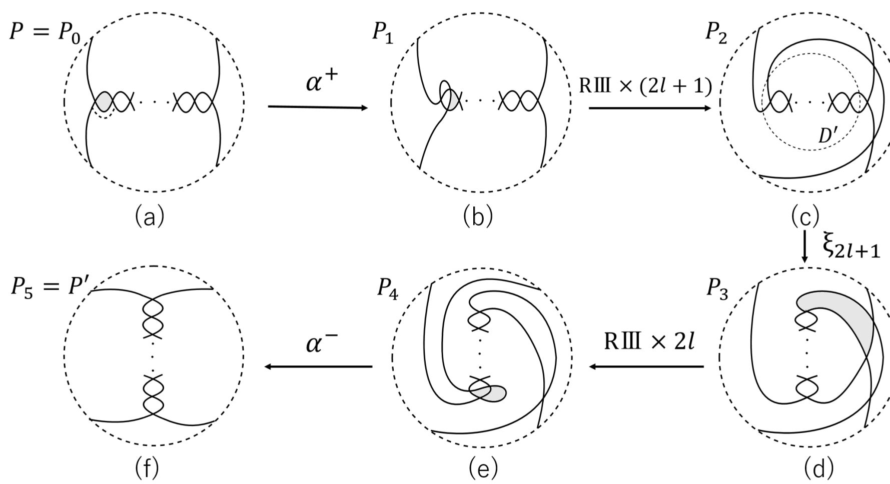


FIGURE 29. Deformation of type ξ_{2l+3}

These complete the proof of Fact 3.1. □

For the statement of the next proposition, we prepare some notations. We consider the 12 types of curves in a disk depicted in Figure 30.

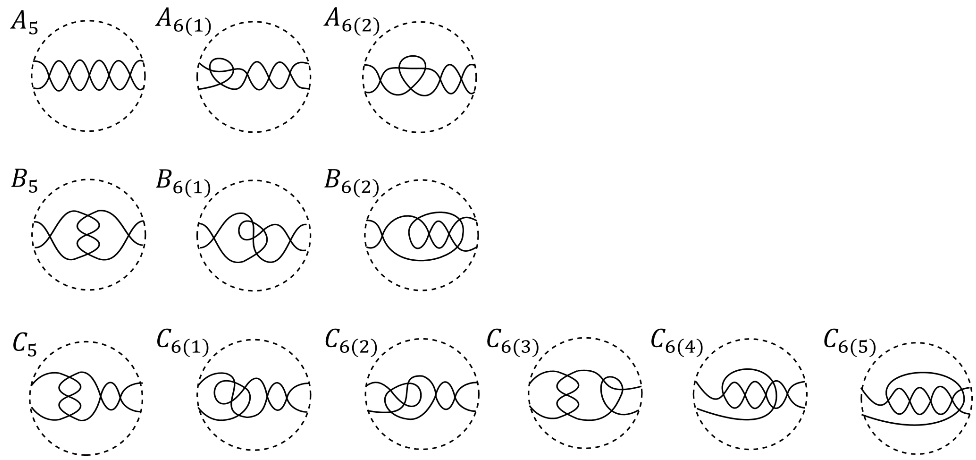
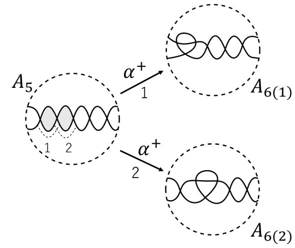


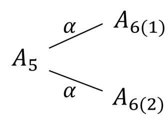
FIGURE 30. The 12 types of curves in a disk

Proposition 3.1. *Let P be a spherical curve. Suppose that there is a disk D in S^2 such that $P \cap D$ is as in Figure 30 A_5. Suppose that a spherical curve P' is obtained from P by a sequence of deformations each of which is either of type RIII or type α performed within D under the constraint such that the number of the double points in D of each of the spherical curves is at most 6. Then $P' \cap D$ is one of the configulations in Figure 30 up to the 4-fold symmetry generated by the vartical reflection and the horizonal reflection of the disk.*

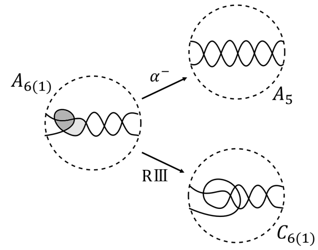
Proof. Let us start with the spherical curve A_5. Since no 1-gon, or triangle is observed there, the possible deformations are all of type α^+, and it is easy to see that they are depicted by the two 2-gons with two dotted arcs named 1 and 2 in Figure 31 A_5(a) up to the 4-fold symmetry. Note that we obtain the spherical curve whose restriction in D is $A_{6(1)}$ in Figure 30 if we perform the deformation along the dotted arc 1, and we obtain the spherical curve whose restriction in D is $A_{6(2)}$ in Figure 30 if we perform the deformation along the dotted arc 2. See Figure 31 A_5(a). We shall describe this fact by using the diagram as in Figure 31 A_5(b). We make similar analyses and diagrams by starting with the configulations $A_{6(1)}$, $A_{6(2)}$, B_5, $B_{6(1)}$, $B_{6(2)}$, C_5, $C_{6(1)}$, $C_{6(2)}$, $C_{6(3)}$, $C_{6(4)}$, $C_{6(5)}$ to obtain Figure 31 $A_{6(1)}$(a)$\sim$ Figure 31 $C_{6(5)}$(b).



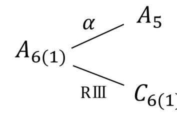
A_5(a)



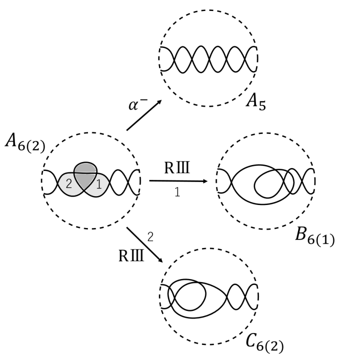
A_5(b)



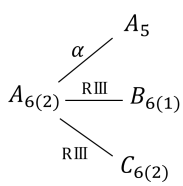
$A_{6(1)}$(a)



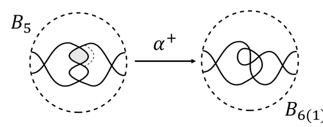
$A_{6(1)}$(b)



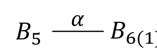
$A_{6(2)}$(a)



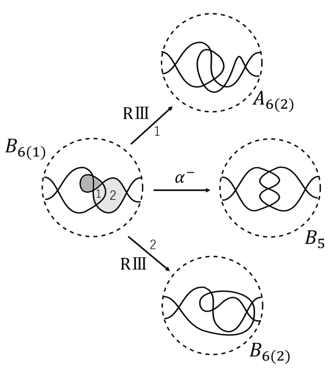
$A_{6(2)}$(b)



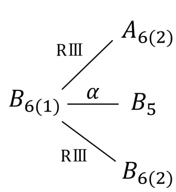
B_5(a)



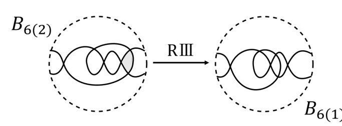
B_5(b)



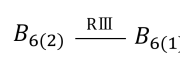
$B_{6(1)}$(a)



$B_{6(1)}$(b)



$B_{6(2)}$(a)



$B_{6(2)}$(b)

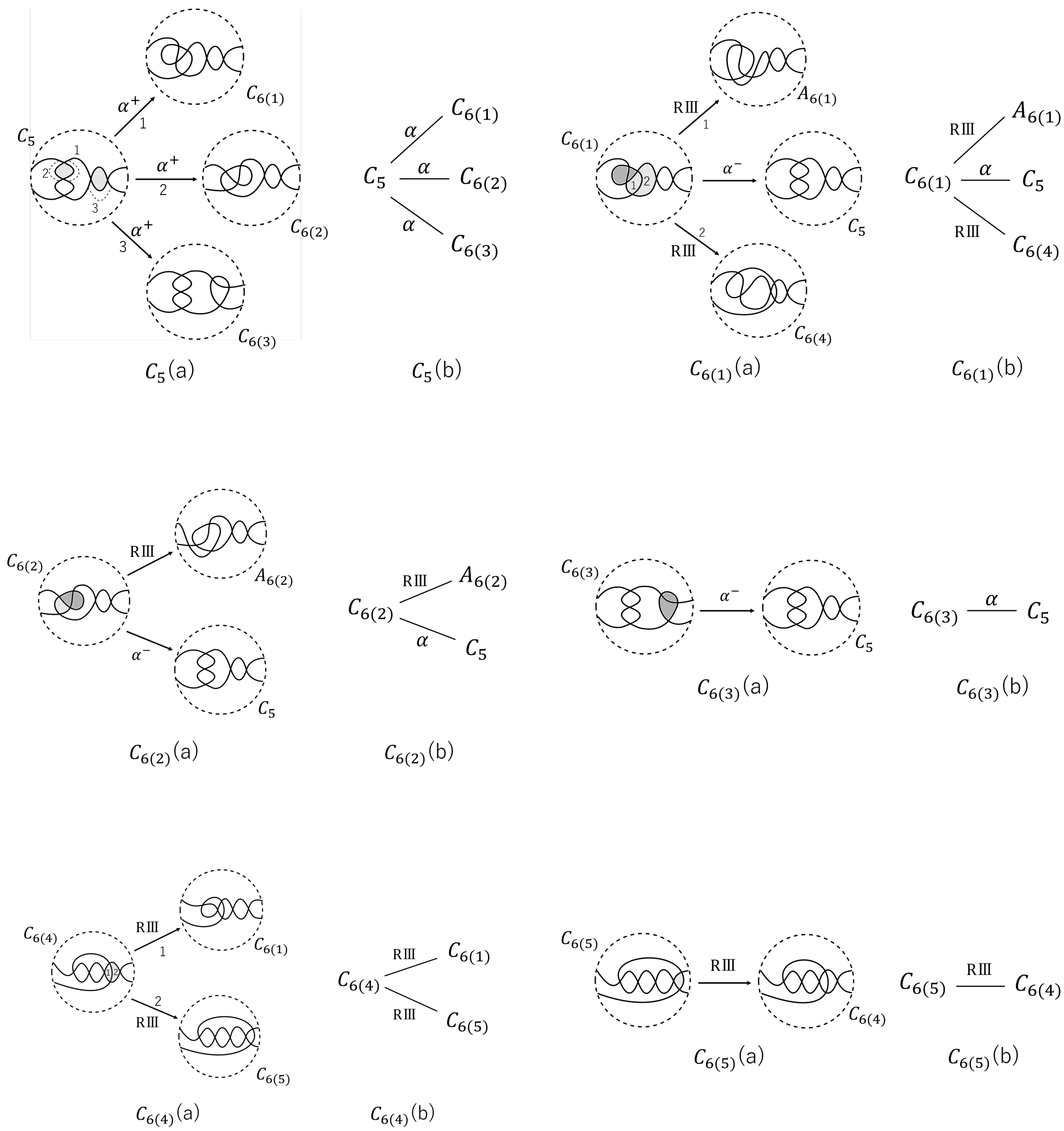


FIGURE 31. Analysis and diagrams

Then we take the graph whose edges are labeled by RIII and α in Figure 32. We note that each vertex corresponds to the configulation in Figure 30 with the same label. We further note that for each vertex X, the subgraph consisting of the edges adjacent to X, and the other vertices of the edges is exactly X(b) in Figure 31. This fact clearly implies Proposition 3.1.

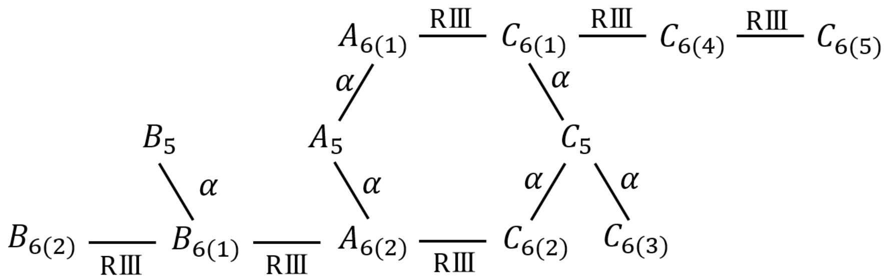


FIGURE 32. The graph whose edges are labeled

□

As an immediate consequence of Proposition 3.1, we obtain the next corollary.

Corollary 3.1. *The deformation of type ξ_5 within a disk D can not be realized by a sequence of deformations each of which is either of type RIII or type α if we pose the following conditions.*

(1) *Every deformation is performed within D, and*
(2) *the numbers of the double points of the spherical curves within D are at most 6.*

3.4. **Pretzel spherical curves.** For an m-tuple of positive integers $a_1, a_2, \ldots, a_m$ ($m \geq 3$), let $P(a_1, a_2, \ldots, a_m)$ be a union of curves depicted in Figure 33.

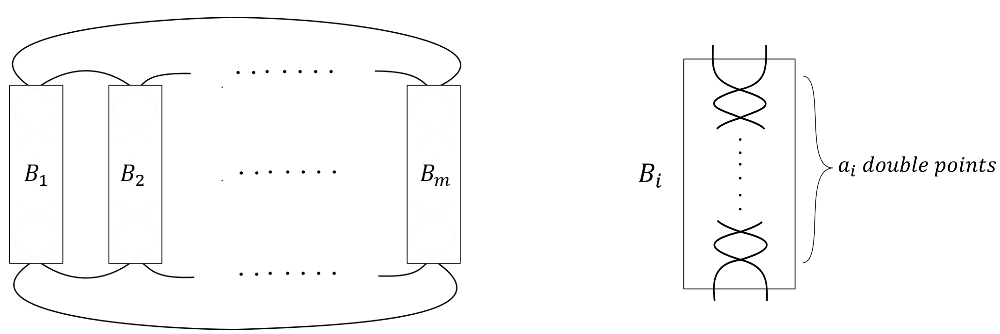


FIGURE 33. $P(a_1, a_2, \ldots, a_m)$

Note that if either one of the following 2 conditions is satisfied, then $P(a_1, a_2, \ldots, a_m)$ is a spherical curve.

(1) Exactly one of $a_1, a_2, \ldots, a_m$ is an even number.
(2) m is an odd number, and every a_i is an odd number.

We call such spherical curve $P(a_1, a_2, \dots, a_m)$ a *pretzel spherical curve* of type $(a_1, a_2, \dots, a_m)$.

In [6], it is shown that every pretzel spherical curve is RI, RIII-equivalent to a trivial spherical curve. In this paper, we study the stable double point numbers between certain pretzel spherical curves and trivial spherical curves.

For positive odd integers $m(\geq 3)$ and p, $P(m;p)$ denotes the pretzel spherical curve with m boxes $B_1, \dots, B_m$, where each B_i contains p double points. Note that $d([P(m;p)]) = mp$. Suppose that $p \geq 3$. By applying deformations of type ξ_p m times, we see that $P(m;p)$ is RI, RIII-equivalent to the 2-bridge spherical curve $C(mp)$. Further by Fact 3.1, we see that

$$sd(P(m;p), C(mp)) \leq mp + (p-1)/2.$$

On the other hand, we note that

$$sd(C(mp), \bigcirc) \leq mp + 1$$

by Theorem 3.1. We note that these arguments have established the following proposition.

Proposition 3.2. *Let $P(m;p)$ be as above. Then*

$$sd(P(m;p), \bigcirc) \leq d([P(m;p)]) + (p-1)/2.$$

We have the impression that the inequality in Proposition 3.2 can be replaced with the equality. That is, it seems that the following equality holds.

$$sd(P(m;p), \bigcirc) = d([P(m;p)]) + (p-1)/2.$$

The next theorem shows that the equality actually holds for the case of $p = 5$.

Theorem 3.2. *Let $P(m;p)$ be as above. Then for each odd number $m(\geq 5)$, we have:*

$$sd(P(m;5), \bigcirc) = d([P(m;5)]) + 2.$$

Proof. We will prove Theorem 3.2 for the case of $m = 5$. The reader will see that arguments work for the cases $m > 5$.

Let B_1, B_2, B_3, B_4, B_5 be the boxes in S^2 as in the definition of $P(5,5,5,5,5) = P(5;5)$. By Proposition 3.2, we see that $sd(P(5;5), \bigcirc) \leq 27$. Hence it is enough to show that $sd(P(5;5), \bigcirc) > 26$ for a proof of Theorem 3.2. Then assume for a contradiction, that $sd(P(5;5), \bigcirc) \leq 26$, and let

$$P(5;5) = P_0 \xrightarrow{op_1} P_1 \xrightarrow{op_2} \cdots \xrightarrow{op_n} P_n = \bigcirc.$$

be a sequence realizing the inequality, which has the shortest length among all such sequences. Note that each op_i is either a deformation of type RI or a deformation of type RIII. Let $\mathcal{B} = B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5$.

Let $\{op_{i_1}, op_{i_2}, \dots, op_{i_s} | i_1 < i_2 < \cdots < i_s\}$ be the subset of the deformations in the sequence

$$P_0 \xrightarrow{op_1} P_1 \xrightarrow{op_2} \cdots \xrightarrow{op_n} P_n$$

consisting of deformations of type RIII. (Hence if $i_t < j < i_{t+1}$ for some t, then op_j is a deformation of type RI.) Then by Proposition 2.1, we may suppose that reduced$(P_{i_{t+1}})$ $(t = 0, 1, 2, \dots, s-1)$ is obtained from reduced(P_{i_t}) by a deformaton of type RIII, type α or type β, where we let $P_{i_0} = P_0$. We have obtained

$$P_0 = \text{reduced}(P_{i_0}) \xrightarrow{\widetilde{op_1}} \text{reduced}(P_{i_1}) \xrightarrow{\widetilde{op_2}} \cdots \xrightarrow{\widetilde{op_s}} \text{reduced}(P_{i_s}) = \bigcirc$$

where each $\widetilde{op_k}$ $(k = 1, 2, \dots, s)$ is a deformation of type RIII, type α or type β. Here we note that $d([P_{i_t}])$ = the number of double points of reduced(P_{i_t}) (Remark 2.1). Then by putting $\widetilde{P_t}$ = reduced(P_{i_t}), we have pbtained a sequence

$$\widetilde{P_0}(= P_0) \xrightarrow{\widetilde{op_1}} \widetilde{P_1} \xrightarrow{\widetilde{op_2}} \cdots \xrightarrow{\widetilde{op_s}} \widetilde{P_s} = \bigcirc$$

such that each $\widetilde{op_k}$ $(k = 1, 2, \dots, s)$ is a deformation of type RIII, type α or type β, and that the number of double points of $\widetilde{P_t}$ ia at most 26.

Claim 3.2. *$\widetilde{op_1}$ is a deformation of type α^+ performed within $\mathcal{B}$.*

Proof of Claim 3.2. Since each region of $\widetilde{P_0}(= P_0 = P(5; 5))$ is not a triangle, $\widetilde{op_1}$ is not of type RIII, α^- or β^-. Since the number of double points of $\widetilde{P_1}$ is at most 26, $\widetilde{op_1}$ is not of type β^+. Hence $\widetilde{op_1}$ is of type α^+. Since every 2-gon region of $\widetilde{P_0}$ is contained in $\mathcal{B}$, $\widetilde{op_1}$ is performed within $\mathcal{B}$. □

Let $\widetilde{op_l}$ be the first deformation which is not performed within $\mathcal{B}$. (Hence $\widetilde{op_1}, \dots, \widetilde{op_{l-1}}$ is performed within $\mathcal{B}$.)

Claim 3.3. *Each of $\widetilde{op_1}, \dots, \widetilde{op_{l-1}}$ is not a deformations of type β.*

Proof of Claim 3.3. Suppose for a contradicton that there exists $k(\leq l-1)$ such that $\widetilde{op_k}$ is of type β. By retaking k, if necessary, we may suppose $\widetilde{op_1}, \dots, \widetilde{op_{k-1}}$ is not of type β (hence, each $\widetilde{op_j}$ $(j = 1, \dots, k-1)$ is of type RIII or type α). Then by Proposition 3.1, we see that each $\widetilde{P_{k-1}} \cap B_i$ $(i = 1, 2, 3, 4, 5)$ is one of the configulations in Figure 30. This shows that the number of the double points of $\widetilde{P_{k-1}}$ is at least 25. If $\widetilde{op_k}$ is β^+, then this implies that the number of double points of $\widetilde{P_k}$ is at least 28, a contradiction. On the other hand, by Figure 30 it is easy to see that there is no β-realm in $\widetilde{P_{k-1}}$, hence $\widetilde{op_k}$ cannot be of type β^-. □

By Claim 3.3 and Proposition 3.1, we see that each $\widetilde{P_{l-1}} \cap B_i$ $(i = 1, 2, 3, 4, 5)$ is one of the configulations in Figure 30. By Proposition 2.2, we may suppose that

$\widetilde{P_{l-1}} \setminus \mathcal{B} = \widetilde{P_0} \setminus \mathcal{B}(= P(5;5) \setminus \mathcal{B})$. We take the points p_N, p_S, p_1, p_2, p_3, p_4, p_5 as in Figure 34.

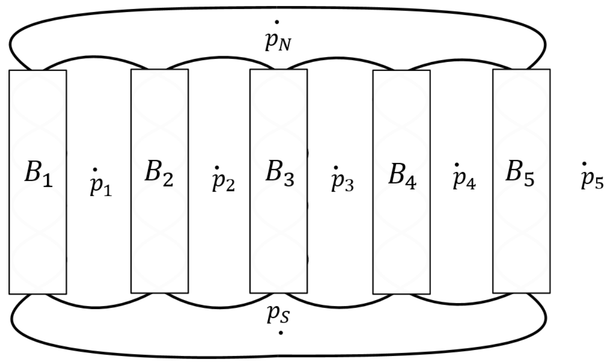


FIGURE 34. The points $p_N, p_S, p_1, p_2, p_3, p_4, p_5$

Then let $R_N, R_S, R_1, R_2, R_3, R_4, R_5$ be the regions of $\widetilde{P_{l-1}}$ containing the points p_N, p_S, p_1, p_2, p_3, p_4, p_5 respectively. Since each $\widetilde{P_{l-1}}$ is RI-minimal, and the region(s) relevant to $\widetilde{op_l}$ is not contained in $\mathcal{B}$, we see that some of the 7 regions is either a 2-gon or a triangle.

Case A: Either R_N or R_S is a 2-gon or a triangle.

Without loss of generality, we may suppose that R_N is a 2-gon or a triangle. Since $m = 5$, we see that there exists s $(1 \leq s \leq 5)$ such that $B_s \cap R_N$ does not contain a corner of R_N, and this shows that there is an isolated arc in $\widetilde{P_{l-1}} \cap B_s$. See Figure 35. But this contradicts Proposition 3.1.

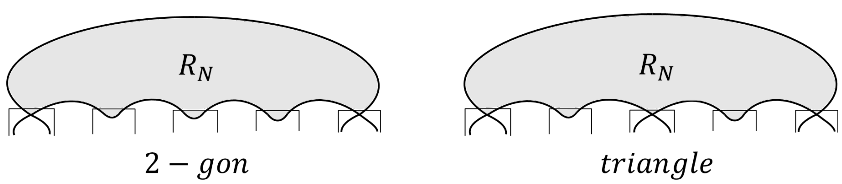


FIGURE 35. R_N is a 2-gon or triangle

Case B: Either one of $R_1, \ldots, R_5$ is a 2-gon or a triangle.

Without loss of generality, we may suppose that R_1 is either a 2-gon or a triangle.

Case B-1: R_1 is a 2-gon.

In this case, since $\widetilde{P_{l-1}} \cap (B_1 \cup B_2)$ does not contain an isolated arc, we see that each of $R_1 \cap B_1$ and $R_1 \cap B_2$ contains exactly one corner of R_1. See Figure 36. Then by Proposition 3.1, we see that the configulations of $\widetilde{P_{l-1}} \cap B_1$ and $\widetilde{P_{l-1}} \cap B_2$ are of type $C_{6(5)}$ in Figure 30. Recall that each of the configulations of $\widetilde{P_{l-1}} \cap B_i$ $(1 \leq i \leq 5)$ is one of the types of Figure 30, and this shows that the number of the double points of $\widetilde{P_{l-1}} \cap B_i$ is greater than or equal to 5, particularly, we note that the number of the double points of type $C_{6(5)}$ configulation is 6. These imply that the number of the double points of $\widetilde{P_{l-1}}$ is greater than 26, a contradiction.

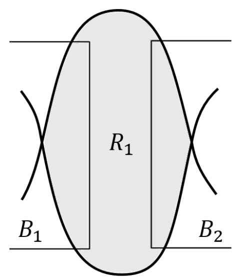


FIGURE 36. In case that R_1 is a 2-gon

Case B-2: R_1 is a triangle.

Since $\widetilde{P_{k-1}} \cap (B_1 \cup B_2)$ does not contain an isolated arc, we may suppose, without loss of generality, that $R_1 \cap B_1$ contains one corner of R_1, and $R_1 \cap B_2$ contains two corners of R_1. See Figure 37.

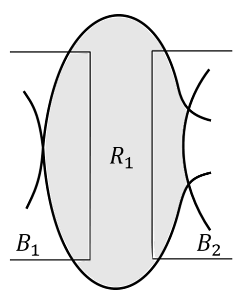


FIGURE 37. In case that R_1 is a triangle

Then by Proposition 3.1, we see that the cofigulation of $\widetilde{P_{l-1}} \cap B_1$ is of type $C_{6(5)}$, and the configulation of $\widetilde{P_{l-1}} \cap B_2$ is of type $B_{6(2)}$, $C_{6(3)}$, $C_{6(4)}$ or $C_{6(5)}$ in Figure 30. Note that all of the numbers of the double points of the configulations of type $B_{6(2)}$, $C_{6(3)}$, $C_{6(4)}$, $C_{6(5)}$ are 6, and this shows that the number of the double points of $\widetilde{P_{l-1}}$ is greater than 26, a contradiction.

These complete the proof of Theorem 3.2 for the case of $m = 5$. It is easy to see that the above arguments work for general m. □

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Department of Mathematics, Nara Women's University Kitauoya Nishimachi, Nara 630-8506, Japan