

# A NEW PROOF OF THE KAC-KAZHDAN CONJECTURE

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Let  $\bar{\mathfrak{g}}$  be a complex simple Lie algebra of rank  $\ell$ ,  $\mathfrak{g}$  the affine Lie algebra associated with  $\bar{\mathfrak{g}}$ :

$$\mathfrak{g} = \bar{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D,$$

where  $K$  is the central element and  $D$  is the degree operator.

A weight  $\lambda$  of  $\mathfrak{g}$  is called a *critical weight* if  $\langle \lambda + \rho, K \rangle = 0$ . A critical weight  $\lambda$  is called *generic* if

$$\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{>0} \text{ for all } \alpha \in \Delta_+^{\text{re}},$$

where  $\Delta_+^{\text{re}}$  is the set of positive real roots of  $\mathfrak{g}$ .

Let  $L(\lambda)$  be the irreducible  $\mathfrak{g}$ -module of highest weight  $\lambda$ .

**Theorem 1.** *If  $\lambda$  is a generic critical weight, then  $\text{ch } L(\lambda) = e^\lambda \prod_{\alpha \in \Delta_+^{\text{re}}} (1 - e^{-\alpha})^{-1}$ .*

Theorem 1 was conjectured by V. Kac and D. Kazhdan [KK]. It has been proved by different methods: for  $\widehat{\mathfrak{sl}}_2$  by M. Wakimoto [Wk], N. Wallach [Wl]; for the classical type affine Lie algebras by T. Hayashi [H] and R. Goodman and N. Wallach [GW]; for general affine Lie algebras by B. Feigin and E. Frenkel [FF1]<sup>1</sup>, J. M. Ku [Ku] and G. Kuroki [Kr]. See also the results in finite characteristic by O. Mathieu [M]; for affine Lie superalgebras by M. Gorelik [G].

The purpose of this note is to give a yet another proof of Theorem 1. We show that Theorem 1 easily follows as a corollary of a general result [A2] on the quantized Drinfeld-Sokolov reduction [FF2, FKW].

**1.1. Preliminaries.** Let  $\bar{\mathfrak{g}} = \bar{\mathfrak{n}}_- \oplus \bar{\mathfrak{h}} \oplus \bar{\mathfrak{n}}_+$  be a triangular decomposition  $\bar{\mathfrak{g}}$ ,  $\Delta$  the set of roots of  $\bar{\mathfrak{g}}$ ,  $\Delta_+$  the set of positive roots of  $\bar{\mathfrak{g}}$ . We also fix roots vectors  $x_\alpha \in \bar{\mathfrak{g}}_\alpha$  with  $\alpha \in \Delta$ .

The Lie algebra  $\bar{\mathfrak{g}}$  is naturally identified with  $\bar{\mathfrak{g}} \otimes \mathbb{C} \subset \mathfrak{g}$ . Then

$$\mathfrak{h} = \bar{\mathfrak{h}} \oplus \mathbb{C}K \oplus \mathbb{C}D$$

is the standard Cartan subalgebra of  $\mathfrak{g}$ . We have  $\mathfrak{h}^* = \bar{\mathfrak{h}}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$ , where  $\Lambda_0$  and  $\delta$  are dual elements of  $K$  and  $D$ , respectively. Let  $\rho = \bar{\rho} + h^\vee \Lambda_0 \in \mathfrak{h}^*$ . Here  $\bar{\rho} = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha \in \bar{\mathfrak{h}}^*$  and  $h^\vee$  is the dual Coxeter number of  $\bar{\mathfrak{g}}$ . Let  $\Delta_+$  be the set of positive roots of  $\mathfrak{g}$ . Then  $\Delta_+ = \Delta_+^{\text{re}} \sqcup \Delta_+^{\text{im}}$ , where  $\Delta_+^{\text{im}} = \{n\delta; n \in \mathbb{Z}_{>0}\}$  is the set of positive imaginary roots. The normalized invariant inner product  $(\ , \ )$  of  $\bar{\mathfrak{g}}$  naturally extends to  $\mathfrak{g}$ .

Let  $M(\lambda)$  be the Verma module of highest weight  $\lambda$ . Then  $L(\lambda)$  is the unique irreducible quotient of  $M(\lambda)$ .

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<sup>1</sup>The details of the proof are given in [Fr, Theorem 4.8]

**Theorem 2** ([KK]). *Let  $\lambda, \mu \in \mathfrak{h}^*$ . Then the following are equivalent:*

- (i)  $L(\mu)$  appears as a subquotient of  $M(\lambda)$ ;
- (ii) *There exists a sequence of positive roots  $\{\beta_k\}_{k=1}^r$ , a sequence of positive integers  $\{n_k\}_{k=1}^r$  and a sequence of weights  $\{\lambda_k\}_{k=1}^r$  such that  $\lambda_0 = \lambda$ ,  $\lambda_r = \mu$  and  $\lambda_k = \lambda_{k-1} - n_k \beta_k$  with  $2(\beta_k, \lambda_{k-1} + \rho) = n_k(\beta_k, \beta_k)$  for  $k = 1, \dots, r$ .*

The following is easy to see.

**Lemma 3.** *A critical weight  $\lambda$  is generic if and only if  $\lambda + n\delta$  is a generic critical weight for every  $n \in \mathbb{Z}$ .*

By Theorem 2 and Lemma 3, we have the following assertion.

**Proposition 4.** *Let  $\lambda$  be a generic critical weight and  $\mu \in \mathfrak{h}^*$ . Then the following are equivalent:*

- (i)  $L(\mu)$  appears as a subquotient of  $M(\lambda)$ ;
- (ii)  $\mu = \lambda - n\delta$  for some  $n \in \mathbb{Z}_{\geq 0}$ .

Let  $\mathcal{O}_{-h^\vee}$  be the category  $\mathcal{O}$  ([K]) of  $\mathfrak{g}$  at the critical level  $-h^\vee$ . For an object  $V$  of  $\mathcal{O}_{-h^\vee}$ , let  $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V^\lambda$  be the weight space decomposition. The formal character  $\text{ch } V$  of  $V$  is defined as  $\text{ch } V = \sum_{\lambda} (\dim_{\mathbb{C}} V^\lambda) e^\lambda$ .

Let  $\lambda$  be a generic critical weight. Then by Lemma 3 and Proposition 4 we have

$$(1) \quad \text{ch } L(\lambda) = \sum_{n \in \mathbb{Z}_{\geq 0}} c_{\lambda, n} \text{ch } M(\lambda - n\delta)$$

with some  $c_{\lambda, n} \in \mathbb{Z}$ . Since

$$\text{ch } M(\mu) = e^\mu \prod_{\alpha \in \Delta_+^{\text{re}}} (1 - e^{-\alpha})^{-1} \prod_{i \geq 1} (1 - q^{-i})^{-\ell},$$

where  $q = e^\delta$ , one has

$$(2) \quad \text{ch } L(\lambda) = e^\lambda \prod_{\alpha \in \Delta_+^{\text{re}}} (1 - e^{-\alpha})^{-1} \sum_{n \in \mathbb{Z}_{\geq 0}} c_{\lambda, n} q^{-n} \prod_{i \geq 1} (1 - q^{-i})^{-\ell}.$$

**1.2. A result from [A2].** Let

$$H^i(V) = H^{\frac{\infty}{2} + i}(L\bar{\mathfrak{n}}_-, V \otimes \mathbb{C}_\chi) \text{ with } i \in \mathbb{Z}$$

for an object  $V$  of  $\mathcal{O}_{-h^\vee}$ . Here,  $L\bar{\mathfrak{n}}_- = \bar{\mathfrak{n}}_- \otimes \mathbb{C}[t, t^{-1}] \subset \mathfrak{g}$  and  $\chi$  is the character of  $L\bar{\mathfrak{n}}_-$  defined by

$$\chi(x_{-\alpha} \otimes t^n) = \begin{cases} 1 & \text{if } n = 0 \text{ and } \alpha \text{ is simple} \\ 0 & \text{otherwise.} \end{cases}$$

Also,  $H^{\frac{\infty}{2} + \bullet}(L\bar{\mathfrak{n}}_-, V)$  is the semi-infinite  $L\bar{\mathfrak{n}}_-$ -cohomology with coefficient in the  $L\bar{\mathfrak{n}}_-$ -module  $V$  ([Fr]). The degree operator  $D$  naturally acts on the space  $H^i(V)$  and one has

$$H^i(V) = \bigoplus_{d \in \mathbb{C}} H^i(V)_d,$$

where  $H^i(V)_d = \{v \in H^i(V); Dv = dv\}$  (see [A2] for details). Set

$$\text{ch } H^i(V) = \sum_{d \in \mathbb{C}} (\dim_{\mathbb{C}} H^i(V)_d) q^d$$

whenever it is well-defined.

**Proposition 5.** *We have  $\text{ch } H^0(M(\lambda)) = q^{\lambda(\mathbf{D})} \prod_{i \geq 1} (1 - q^{-i})^{-\ell}$  for all  $\lambda$ .*

Proposition 5 is essentially proved in [FB] based on the idea of [dBT] (see [A1, Remark 5,8]).

By [A2, Main Theorem 1] we have the following assertion.

**Theorem 6.**

- (i) *We have  $H^i(V) = \{0\}$  with  $i \neq 0$  for all objects  $V$  of  $\mathcal{O}_{-h^\vee}$ .*
- (ii) *Let  $\lambda$  be a critical weight. If the classical part of  $\lambda$  is anti-dominant, then*

$$\text{ch } H^0(L(\lambda)) = q^{\lambda(\mathbf{D})}.$$

*Otherwise  $H^0(L(\lambda)) = \{0\}$ .*

Here, in Theorem 6, the classical part  $\bar{\lambda}$  of  $\lambda \in \mathfrak{h}^*$  is the image of the projection  $\mathfrak{h}^* \rightarrow \bar{\mathfrak{h}}^*$ , and it is called anti-dominant if  $\langle \bar{\lambda} + \bar{\rho}, \alpha^\vee \rangle \notin \mathbb{Z}_{>0}$  for all  $\alpha \in \Delta_+$ . In particular, the classical part of a generic critical weight is anti-dominant.

**1.3. Proof of Theorem 1.** From (1) and Theorem 6 (i), it follows that

$$\text{ch } H^0(L(\lambda)) = \sum_{n \in \mathbb{Z}_{\geq 0}} c_{\lambda, n} \text{ch } H^0(M(\lambda - n\delta))$$

Thus by Proposition 5 and Theorem 6 (ii), (iii) one obtains

$$q^{\lambda(\mathbf{D})} = q^{\lambda(\mathbf{D})} \sum_{n \in \mathbb{Z}_{\geq 0}} c_{\lambda, n} q^{-n} \prod_{i \geq 1} (1 - q^{-i})^{-\ell}.$$

Thus

$$\sum_{n \in \mathbb{Z}_{\geq 0}} c_{\lambda, n} q^{-n} \prod_{j \geq 1} (1 - q^{-j})^{-\ell} = 1.$$

But then the theorem follows from (2).  $\square$

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