

Trace formula and trace identity of twisted Hecke operators on the spaces of cusp forms of weight $k + 1/2$ and level $32M$

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Abstract: Let M be an odd positive integer, χ an even quadratic character defined modulo $32M$, and ψ a quadratic primitive character of conductor divisible by 8. Then, we can define twisted Hecke operators $R_\psi \tilde{T}(n^2)$ on the space of cusp forms of weight $k + 1/2$, level $32M$, and character χ , under certain conditions on the conductors of χ and ψ . This is a specific feature of the case of half-integral weight. We give explicit trace formulas of the twisted Hecke operators and their trace identities.

Key words: Trace formula; twisting operator; half-integral weight; trace identity; Hecke operator; cusp form.

1. Introduction. Let k and N be positive integers with $4 \mid N$ and χ an even quadratic Dirichlet character defined modulo N . We denote the space of cusp forms of weight $k + 1/2$, level N , and character χ by $S(k + 1/2, N, \chi)$. Let R_ψ be the twisting operator for a quadratic primitive character ψ and $\tilde{T}(n^2)$ the n^2 -th Hecke operator of weight $k + 1/2$. In the previous papers [U3] and [U4], we reported trace formulas and trace identities of the twisted Hecke operators $R_\psi \tilde{T}(n^2)$ on $S(k + 1/2, N, \chi)$ for various cases. However, we missed one peculiar case of level $32M$ in those papers. Let ψ be a quadratic primitive character whose conductor is divisible by 8. Then $S(k + 1/2, 32M, \chi)$ is not generally closed under R_ψ . But, under certain conditions of χ and ψ , R_ψ defines a linear operator on $S(k + 1/2, 32M, \chi)$ (See Proposition 1 below). This phenomenon is specific to modular forms of half-integral weight. The aim of this paper is to report explicit trace formulas and trace identities in this case. The details will appear in [U5] or another.

2. Notation. We use the same notation as in the previous paper [U1]. See [U1] and [U2] for the details of notation. Here we explain some of symbols for convenience.

Let k , N , and χ be the same as above. Let a be a non-zero integer and b a positive integer. We write $a \mid b^\infty$ if every prime factor of a divides b .

We denote by $(:)$ the Kronecker symbol. See

[M, p.82] for a definition of this symbol.

Let \mathbf{H} be the complex upper half-plane. Put $j(\gamma, z) = (-\frac{1}{d})^{-1/2} (\frac{c}{d})(cz + d)^{1/2}$ for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$ and $z \in \mathbf{H}$. Let $\mathfrak{G}(k + 1/2)$ be the covering group of $GL_2^+(\mathbf{R})$ (cf. [U1, §0(c)]). For a complex-valued function f on \mathbf{H} and $(\alpha, \phi) \in \mathfrak{G}(k + 1/2)$, we define a function $f|(\alpha, \phi)$ on \mathbf{H} by: $f|(\alpha, \phi)(z) = \phi(z)^{-1} f(\alpha z)$. By $\Delta_0(N, \chi) = \Delta_0(N, \chi)_{k+1/2}$, we denote the subgroup of $\mathfrak{G}(k + 1/2)$ consisting of all pairs $\gamma^* := (\gamma, \phi)$, where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and $\phi(z) = \chi(d)j(\gamma, z)^{k+1/2}$.

Let ρ be any Dirichlet character. We denote the conductor of ρ by $f(\rho)$ and for any prime number p , the p -primary component of ρ by ρ_p . Furthermore we set $\rho_A := \prod_{p \mid A} \rho_p$ for an arbitrary positive integer A . Here p runs over all prime divisors of A .

Let V be a finite-dimensional vector space over \mathbf{C} . We denote the trace of a linear operator T on V by $\text{tr}(T; V)$.

3. Twisting operator. From now on, we assume that $N = 32M$ with an odd positive integer M and that $f(\chi_2)$ divides 4.

Let ψ be a quadratic primitive character of conductor r such that the conductor of ψ_2 is equal to 8. Then we can express the conductor r as $r = 8L$ with a squarefree odd positive integer L .

From now on until the end of this paper, we assume the condition $L^2 \mid M$.

Proposition 1. *Under the above assumptions, the twisting operator R_ψ for ψ*

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$$f = \sum_{n \geq 1} a(n) q^n \mapsto f \mid R_\psi := \sum_{n \geq 1} a(n) \psi(n) q^n$$

$$(q := \exp(2\pi\sqrt{-1}z), z \in \mathbf{H})$$

defines a linear operator of $S(k+1/2, 32M, \chi)$.

Proof. Take any $f \in S(k+1/2, 32M, \chi)$. From $S(k+1/2, 32M, \chi) \subset S(k+1/2, 64M, \chi)$ and [Sh, Lemma 3.6], we see that $f \mid R_\psi \in S(k+1/2, 64M, \chi)$. Since any element of $S(k+1/2, 64M, \chi)$ is fixed by all elements of $\Delta_0(64M, \chi)$, it is sufficient for checking the statement to show that $f \mid R_\psi$ is fixed by the representative $\begin{pmatrix} 1 & 0 \\ 32M & 1 \end{pmatrix}^*$ of $\Delta_0(32M, \chi)/\Delta_0(64M, \chi)$.

Now, we put

$$\mathfrak{g}(\psi) := \sum_{i \bmod r} \psi(i) \exp(2\pi\sqrt{-1}i/r)$$

and

$$\xi(u) := \left(\begin{pmatrix} r & u \\ 0 & r \end{pmatrix}, 1 \right) \in \mathfrak{G}(k+1/2)$$

for any integer u .

Observing that $\bar{\psi} = \psi$ (because ψ is quadratic), we can express R_ψ as follows (cf. [Sh, Lemma 3.6]):

$$\mathfrak{g}(\psi) f \mid R_\psi = \sum_{\substack{u \bmod r \\ (u,r)=1}} \psi(u) f \mid \xi(u).$$

Hence

$$(1) \quad \mathfrak{g}(\psi) f \mid R_\psi \begin{pmatrix} 1 & 0 \\ 32M & 1 \end{pmatrix}^* = \sum_{\substack{u \bmod r \\ (u,r)=1}} \psi(u) f \mid \xi(u) \begin{pmatrix} 1 & 0 \\ 32M & 1 \end{pmatrix}^*.$$

For any $u \in (\mathbf{Z}/r\mathbf{Z})^\times$, take $v \in (\mathbf{Z}/r\mathbf{Z})^\times$ such that

$$(2) \quad \begin{cases} v \equiv u \pmod{L}, \\ v \equiv u+4 \pmod{8}. \end{cases}$$

By straightforward calculation, we have

$$\begin{aligned} \gamma_0 &:= \begin{pmatrix} r & u \\ 0 & r \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 32M & 1 \end{pmatrix} \begin{pmatrix} r & v \\ 0 & r \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1+32Mu/r & r^{-2}(r(u-v)-32Muv) \\ 32M & 1-32Mv/r \end{pmatrix} \\ &\in \Gamma_0(32M) \end{aligned}$$

and also

$$(3) \quad \xi(u) \begin{pmatrix} 1 & 0 \\ 32M & 1 \end{pmatrix}^* \xi(v)^{-1}$$

$$= \left(\gamma_0, (32M(z-v/r)+1)^{k+1/2} \right).$$

Moreover, we can calculate $j(\gamma_0, z)$ as follows: First, since $32Mv/r = 4(M/L)v \equiv 0 \pmod{4}$, we have $\left(\frac{-1}{1-32Mv/r}\right) = 1$. Next, observing that $1-4(M/L)v \equiv 1-4 \equiv 5 \pmod{8}$, we can calculate as follows:

$$\begin{aligned} \left(\frac{32M}{1-32Mv/r} \right) &= \left(\frac{32M}{1-4(M/L)v} \right) \\ &= \left(\frac{2M/L^2}{1-4(M/L)v} \right) \\ &= \left(\frac{2}{1-4(M/L)v} \right) \left(\frac{M/L^2}{1-4(M/L)v} \right) \\ &= \left(\frac{2}{1-4(M/L)v} \right) \left(\frac{M/L^2}{1} \right) \\ &= \left(\frac{2}{1-4(M/L)v} \right) = \left(\frac{2}{5} \right) = -1. \end{aligned}$$

Hence, we have

$$(4) \quad j(\gamma_0, z) = -(32Mz+1-32Mv/r)^{1/2}.$$

On the other hand, since $f(\chi_2) \mid 4$ and $f(\chi_M) \mid \prod_{p \mid M} p \mid (M/L)$, we have

$$\begin{aligned} \chi(1-32Mv/r) &= \chi(1-4(M/L)v) \\ &= \chi_2(1-4(M/L)v) \chi_M(1-4(M/L)v) \\ &= \chi_2(1) \chi_M(1) = 1. \end{aligned}$$

Therefore by (3), (4), and the above, we get

$$(5) \quad \xi(u) \begin{pmatrix} 1 & 0 \\ 32M & 1 \end{pmatrix}^* \xi(v)^{-1} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, -1 \right) \gamma_0^*.$$

Here $\gamma_0^* := (\gamma_0, \chi(1-32Mv/r)j(\gamma_0, z)^{2k+1}) \in \Delta_0(32M, \chi)$. Then, from (1) and (5), we have

$$(6) \quad \begin{aligned} \mathfrak{g}(\psi) f \mid R_\psi \begin{pmatrix} 1 & 0 \\ 32M & 1 \end{pmatrix}^* &= - \sum_{\substack{u \bmod r \\ (u,r)=1}} \psi(u) f \mid \gamma_0^* \xi(v) \\ &= - \sum_{\substack{u \bmod r \\ (u,r)=1}} \psi(u) f \mid \xi(v). \end{aligned}$$

Moreover, we can show from (2)

$$(7) \quad \begin{aligned} \psi(v) &= \psi_2(v) \psi_L(v) = \psi_2(u+4) \psi_L(u) \\ &= -\psi_2(u) \psi_L(u) = -\psi(u). \end{aligned}$$

Since the correspondence $u \mapsto v$ is a permutation of $(\mathbf{Z}/r\mathbf{Z})^\times$, we finally obtain

$$\mathfrak{g}(\psi) f \mid R_\psi \begin{pmatrix} 1 & 0 \\ 32M & 1 \end{pmatrix}^* = \sum_{\substack{v \bmod r \\ (v,r)=1}} \psi(v) f \mid \xi(v)$$

$$= \mathfrak{g}(\psi) f \mid R_\psi.$$

The proof is completed. \square

In the case of $k=1$, we need to make a modification. In this case, the following is well-known (cf. [U1, §0(c)]). The space $S(3/2, 32M, \chi)$ contains a subspace $U(32M; \chi)$ which corresponds to a space of Eisenstein series via the Shimura correspondence. And the subspace $U(32M; \chi)$ is generated by theta series of special type. Let $V(32M; \chi)$ be the orthogonal complement of $U(32M; \chi)$ in $S(3/2, 32M, \chi)$ with respect to the Petersson inner product. Then it is also well-known that $V(32M; \chi)$ corresponds to a space of cusp forms via the Shimura correspondence. Hence we need to consider the subspace $V(32M; \chi)$ in place of $S(3/2, 32M, \chi)$ in the case of $k=1$.

The subspaces $U(32M; \chi)$ and $V(32M; \chi)$ are closed under the twisting operator R_ψ (See [U5] for a proof and refer also to [U2, p.94]). Hence R_ψ gives a linear operator also on the subspace $V(32M; \chi)$. Moreover, the n^2 -th Hecke operators $\tilde{T}(n^2)$, $(n, 32M) = 1$, also define linear operators on the subspace $V(32M; \chi)$ (cf. [U1, p.508]).

Thus for any positive integer n with $(n, 32M) = 1$, we can consider the twisted Hecke operator $R_\psi \tilde{T}(n^2)$ on the spaces $S(k+1/2, 32M, \chi)$ (if $k \geq 2$) and $V(32M; \chi)$ (if $k=1$) (cf. [U2, p.86]).

4. Trace formula. Now we state an explicit trace formula of the twisted Hecke operator $R_\psi \tilde{T}(n^2)$.

Theorem 1. *Let notation and assumptions be the same as above. Let $\tilde{T}(n^2) = \tilde{T}_{k+1/2, 32M, \chi}(n^2)$ be the n^2 -th Hecke operator for a positive integer n with $(n, 32M) = 1$ (cf. [U1, §0(c)]). Then explicit trace formulas of the twisted Hecke operator $R_\psi \tilde{T}(n^2)$ on the spaces $S(k+1/2, 32M, \chi)$ (if $k \geq 2$) and $V(32M; \chi)$ (if $k=1$) are given as follows:*

$$\begin{aligned} \operatorname{tr}(R_\psi \tilde{T}(n^2); S(k+1/2, 32M, \chi)) &= t(p) + t(e), \\ &\quad (\text{if } k \geq 2). \\ \operatorname{tr}(R_\psi \tilde{T}(n^2); V(32M; \chi)) &= t(p) + t(e) + t(d), \\ &\quad (\text{if } k = 1). \end{aligned}$$

Here $t(p)$, $t(e)$, $t(d)$ are the contributions from the parabolic, elliptic, and degree part respectively. They are given by the formulas (1.1)–(1.3) below.

We use the following notation in those formulas. Let \mathbf{Z}_+ be the set of all positive integers. For a real number x , $[x]$ means the greatest integer less

than and equal to x . For a prime number p , let $\operatorname{ord}_p(\cdot)$ be the p -adic additive valuation with $\operatorname{ord}_p(p) = 1$ and $|\cdot|_p$ the p -adic absolute value which is normalized with $|p|_p = p^{-1}$. Put $\nu = \nu_p := \operatorname{ord}_p(M)$ for any odd prime number p . And we decompose the level $N = 32M$ with respect to L as follows:

$$\begin{aligned} N &= 32L_0L_2, \quad L_0 > 0, \quad L_2 > 0, \\ L_0 &\mid L^\infty, \quad (L_2, 2L) = 1. \end{aligned}$$

Then we have $L_2 = N \prod_{p \mid 2L} |N|_p$.

(1.1)

$$\begin{aligned} t(p) &= (-1)^k \psi(-1)^k n^{k-1} \chi(n) \\ &\times \begin{cases} \chi_2(-1), & \text{if } n \equiv \psi(-1) \pmod{4}, \\ 1, & \text{if } n \equiv -\psi(-1) \pmod{4}, \end{cases} \\ &\times \prod_{p \mid L} p^{[(\nu-1)/2]} \\ &\times \prod_{p \mid L_2} \left(p^{[\nu/2]} + \left(\frac{-8Ln}{p} \right)^\nu p^{[(\nu-1)/2]} \right) \\ &\times \sum_{0 < a \mid n_0} h'(-8Ln/a^2). \end{aligned}$$

Here the notation is as follows: we decompose $n = n_0^2 n_1$ ($n_0, n_1 \in \mathbf{Z}_+$, n_1 : squarefree). And let $\mathcal{O}(-d)$ be the order of discriminant $-d$ in the imaginary quadratic number field $\mathbf{Q}(\sqrt{-d})$, $h(-d)$ the number of proper ideal classes of the order $\mathcal{O}(-d)$, and $w(-d)$ a half of the number of units in $\mathcal{O}(-d)$. Then put $h'(-d) := h(-d)/w(-d)$.

(1.2)

$$\begin{aligned} t(e) &= -\psi(-1)^k r^{1-k} \chi_L(-n) \times \prod_{p \mid L} p^{[(\nu-1)/2]} \\ &\times 2 \chi_2(\psi(-1)) \\ &\times \sum_{\substack{0 < s < 2\sqrt{rn} \\ s: (*), s \equiv 0 \pmod{8}}} \pi_k(s, rn) h'(D) \alpha_D(m'_1) \\ &\times \prod_{p \mid L_2} (p^{-\operatorname{ord}_p(s)} n_p(\theta_p)). \end{aligned}$$

Here the condition $(*)$ of s is the following

$$\begin{aligned} (*) \quad \operatorname{ord}_p(s) &\geq [(\nu_p + 1)/2] \\ &\text{for all prime divisors } p \text{ of } L. \end{aligned}$$

The other notation is defined as follows: We decompose $s^2 - 4rn = m_1^2 D$ with $m_1 \in \mathbf{Z}_+$ and a discriminant D of an imaginary quadratic field. We put $m'_1 := m_1 \prod_{p \mid N} |m_1|_p$ and $\theta_p := \operatorname{ord}_p(sm_1)$ for a

prime number p . Moreover we put a constant $\pi_k(s, rn) := (x^{2k-1} - y^{2k-1})/(x - y)$, where x, y are two roots of the quadratic equation $X^2 - sX + rn = 0$. For a positive integer A , we define a constant $\alpha_D(A)$ by

$$\alpha_D(A) := \prod_{q|A} \left\{ (q^{e+1} - 1) - \left(\frac{D}{q} \right) (q^e - 1) \right\} / (q - 1),$$

where $A = \prod_{q|A} q^e$ is the prime decomposition of A . The constant $h'(D)$ is the same as in the parabolic part $t(p)$. Finally, the constants $n_p(\theta_p)$ ($p \mid L_2$) are given by the table below.

Table of $n_p(\theta_p)$.

$$\begin{aligned} & \text{Case (1)} \quad (p \mid L_2 \text{ and } p \mid s) \\ & \chi_p(r) \chi_p(D) \times n_p(\theta_p) \\ & = \begin{cases} p^{\theta_p} \left(p^{[\nu/2]} + \left(\frac{D}{p} \right)^{\nu} p^{[(\nu-1)/2]} \right), & \text{if } \theta_p \geq [(\nu+1)/2]. \\ \left(1 + \left(\frac{D}{p} \right) \right) p^{2\theta_p}, & \text{if } \theta_p \leq [(\nu-1)/2]. \end{cases} \\ & \text{Case (2)} \quad (p \mid L_2, p \nmid s \text{ and } p \mid D) \\ & \chi_p(r) \times n_p(\theta_p) \\ & = \begin{cases} \{ (p^{[\nu/2]} + p^{[(\nu-1)/2]}) p^{\theta_p+1} - (p^{\nu} + p^{\nu-1}) \} (p-1)^{-1}, & \text{if } \theta_p \geq [\nu/2]. \\ 0, & \text{if } \theta_p \leq [\nu/2] - 1. \end{cases} \\ & \text{Case (3)} \quad (p \mid L_2, p \nmid s \text{ and } p \nmid D) \\ & \chi_p(r) \times n_p(\theta_p) \\ & = \begin{cases} \left(p - \left(\frac{D}{p} \right) \right) (p^{[\nu/2]} + p^{[(\nu-1)/2]}) (p^{\theta_p} - p^{[\nu/2]}) \\ \quad \times (p-1)^{-1} + \left(p^{[\nu/2]} + \left(\frac{D}{p} \right)^{\nu} p^{[(\nu-1)/2]} \right) p^{[\nu/2]}, & \text{if } \theta_p \geq [(\nu+1)/2]. \\ \left(1 + \left(\frac{D}{p} \right) \right) p^{2\theta_p}, & \text{if } \theta_p \leq [(\nu-1)/2]. \end{cases} \end{aligned}$$

(1.3)

$$\begin{aligned} t(d) &= \psi(-1) \chi_2(\psi(-1)) \chi_L(-n) \chi_{L_2}(r) \\ &\times \prod_{p|n} \frac{p^{\tau+1} - 1}{p - 1} \\ &\times \prod_{p|L_2} \left\{ \left[\frac{\nu_p - \alpha_p}{2} \right] + 1 + \left[\frac{\nu_p + \alpha_p - 1}{2} \right] \left(\frac{-rn}{p} \right) \right\}. \end{aligned}$$

Here the notation is as follows: Let $n = \prod_{p|n} p^{\tau}$ be

the prime decomposition of n . For any prime divisor p of L_2 , the constant α_p is defined by

$$\chi_p = \left(\frac{-}{p} \right)^{\alpha_p}, \quad (\alpha_p = 0, 1).$$

This is possible, because χ is a quadratic character and p is odd.

5. Trace identity. Using the above explicit trace formula, we can obtain trace identities between the twisted Hecke operators $R_{\psi} \tilde{T}(n^2)$ and linear combinations of Hecke operators of integral weight and Atkin-Lehner involutions.

We prepare a little more notation for the statement of trace identity.

First we put

$$N_0 := \prod_{p|L} p^{2[(\nu_p-1)/2]+1}.$$

Here p runs over all prime divisors of L .

Next, let A be any positive integer. For any odd prime number p and any integers a, b ($0 \leq a \leq \text{ord}_p(A)/2$), we put

$$\begin{aligned} & \lambda_p(\chi_p, \text{ord}_p(A); b, a) \\ & := \begin{cases} 1, & \text{if } a = 0, \\ 1 + \left(\frac{-b}{p} \right), & \text{if } 1 \leq a \leq [(\text{ord}_p(A) - 1)/2], \\ \chi_p(-b), & \text{if } \text{ord}_p(A) \text{ is even} \\ & \text{and } a = \text{ord}_p(A)/2 \geq 1. \end{cases} \end{aligned}$$

Then for any integer b and any square integer c , we put

$$\Lambda_{\chi}(\psi, A; b, c) := \prod_{\substack{p|A \\ (p,r)=1}} \lambda_p(\chi_p, \text{ord}_p(A); b, \text{ord}_p(c)/2).$$

Here p runs over all prime divisors of A which are prime to r .

Furthermore, let B be a positive divisor of A such that $B \mid r^{\infty}$ and $(A/B, B) = 1$. For all positive integers n such that $(n, 32M) = 1$, we define

$$\begin{aligned} & \mathcal{C}_{\psi}[2k, n; A, B, \chi] = \mathcal{C}_{\psi}[A, B, \chi] \\ & := \sum_{\substack{0 < N_1 | A \\ N_1 = \square, (N_1, r) = 1}} \Lambda_{\chi}(\psi, A; rn, N_1) \\ & \quad \times \text{tr}(W(BN_1)T(n); S(2k, N_1N_2)), \end{aligned}$$

where N_1 runs over all square positive divisors of A which are prime to r and $N_2 := A \prod_{p|N_1} |A|_p$. $T(n)$ is the Hecke operator of weight $2k$ and $W(BN_1)$ is the Atkin-Lehner involution. $S(2k, N_1N_2)$ is the space

of cusp forms of weight $2k$ and level N_1N_2 . See [U1, §0(b) and §3] for the details of these definitions.

Remark. All spaces occurring in the definition of $\mathcal{C}_\psi[A, B, \chi]$ are contained in the space $S(2k, A)$.

Finally, $\chi'_r := \prod_{p|N, (p,r)=1} \chi_p$, where p runs over all prime divisors of $N = 32M$ which are prime to r . Then we put

$$c(k, n; \psi, \chi) = c(\psi, \chi) := \psi(-1)^k \chi_r(n) \chi'_r(-r).$$

Under this notation, we can state trace identities of the twisted Hecke operators $R_\psi \tilde{T}(n^2)$.

Theorem 2. *Let notation and assumptions be the same as above. For all positive integers n such that $(n, 32M) = 1$, we have the following trace identity:*

$$\begin{aligned} & \begin{cases} \operatorname{tr}(R_\psi \tilde{T}(n^2); S(k + 1/2, 32M, \chi)) & \text{if } k \geq 2 \\ \operatorname{tr}(R_\psi \tilde{T}(n^2); V(32M; \chi)) & \text{if } k = 1 \end{cases} \\ &= c(\psi, \chi) \times \begin{cases} \chi_2(-1), & \text{if } n \equiv \psi(-1) \pmod{4} \\ 1, & \text{if } n \equiv -\psi(-1) \pmod{4} \end{cases} \\ & \quad \times \mathcal{C}_\psi[8N_0L_2, 8N_0, \chi]. \end{aligned}$$

6. Concluding remark. We can expect to establish a theory of newforms of half-integral weight by using the above trace identities. See [U6] for related results for the case of level 2^m .

References

- [M] T. Miyake, *Modular forms*, Translated from the Japanese by Yoshitaka Maeda, Springer, Berlin, 1989.
- [Sh] G. Shimura, On modular forms of half integral weight, *Ann. of Math. (2)* **97** (1973), 440–481.
- [U1] M. Ueda, The decomposition of the spaces of cusp forms of half-integral weight and trace formula of Hecke operators, *J. Math. Kyoto Univ.* **28** (1988), no. 3, 505–555.
- [U2] M. Ueda, The trace formulae of twisting operators on the spaces of cusp forms of half-integral weight and some trace relations, *Japan. J. Math. (N.S.)* **17** (1991), no. 1, 83–135.
- [U3] M. Ueda, Trace formula of twisting operators of half-integral weight in the case of even conductors, *Proc. Japan Acad. Ser. A Math. Sci.* **79** (2003), no. 4, 85–88.
- [U4] M. Ueda, Trace identities of twisted Hecke operators on the spaces of cusp forms of half-integral weight, *Proc. Japan Acad. Ser. A Math. Sci.* **80** (2004), no. 7, 131–135.
- [U5] M. Ueda, The trace formulae of twisting operators on the spaces of cusp forms of half-integral weight and trace identities II. (in preparation).
- [U6] M. Ueda, Newforms of half-integral weight in the case of level 2^m . (Preprint). <http://www.math.nara-wu.ac.jp/personal/ueda/index.e.html>