Feller property and Dirichlet forms for skew product diffusion processes and their time change

TAKEMURA Tomoko* and TOMISAKI Matsuyo†

1 Introduction

Let $\mathrm{R} = [R_t, P_r^{\mathrm{R}}, \zeta^{\mathrm{R}}]$ be a one dimensional diffusion process (ODDP for brief) on an open interval $I = (l_1, l_2)$, and $\Theta = [\Theta_t, P_\theta^\Theta]$ be the spherical Brownian motion on $S^{d-1} \subset \mathbb{R}^d$. In this article we study Feller property of the skew product $\Xi = [\Xi_t = (R_t, \Theta_{\mathbf{f}(t)}), P_{(r,\theta)}^\Xi = P_r^{\mathrm{R}} \otimes P_\theta^\Theta,\ (r,\theta) \in I \times S^{d-1}, \zeta^\Xi]$ with respect to a positive continuous additive functional (PCAF for brief) $\mathbf{f}(t)$ of the ODDP R. We also study Feller property of time changed processes of the skew product Ξ. Further we present Dirichlet forms of the skew product Ξ and time changed processes.

Fukushima and Oshima [6], and Okura [11] considered the skew product of conservative Markov processes and obtained some expression of Dirichlet form corresponding to the skew product. Z.-Q. Chen and Fukushima [1], and Fukushima [4], [5] showed the Dirichlet forms corresponding to ODDPs. In [13], by using results from [1], [6] and [11], we obtained the Dirichlet form corresponding to Ξ under the assumption that both of l_i, $i = 1,\ 2$, are entrance or natural in the sense of Feller [3]. In this article we remove that assumption, and consider the same problem as in [13]. Okura [12] investigated Dirichlet forms corresponding to the skew product of Markov processes which are not necessarily conservative. He regarded the Revuz measure of PCAF as a killing measure, and considered a new Dirichlet form with killing part. Then he derived Dirichlet forms corresponding to the skew product. Fukushima [5] made clear Dirichlet forms of ODDPs with killing measure. Combining the results of [5] and [12], we can obtain the Dirichlet form corresponding to Ξ in general case (Proposition 2.1). In our argument corresponding to the case that l_i is regular with reflecting boundary condition in the sense of Feller, we assume that the Revuz measure of PCAF is bounded near l_i. This assumption is necessary and sufficient for the Revuz measure of PCAF in some sense. We show this fact following the same argument as for Feller property of the skew product (Remark 3.3). The details on properties of ODDPs are well known (cf. [8], [10]). By using their results, we show Feller properties of the skew product Ξ (Theorem 3.2). Further we show Feller properties and Dirichlet forms of the time change of Ξ following the same argument as in [13] (Theorems 4.2 and 5.1).

2 Dirichlet form of skew product

Let $\rho = [\rho_t, P_r^\rho, \zeta^\rho]$ be a one dimensional generalized diffusion process (ODGDP for brief) on an open interval $I = (l_1, l_2)$ with the scale function s, the speed measure m and the

*Faculty (Natural Sciences, Mathematics) Associate Professor, This work was supported by JSPS KAKENHI Grant-in-Aid for Scientific Research (C) 22K03353
†Nara Women's University, Professor Emeritus

killing measure k, where $0 \le l_1 < l_2 \le \infty$ and ζ^ρ is the life time of ρ. s is a continuous increasing function on I, m is a nontrivial nonnegative Radon measure on I, and k is a nonnegative Radon measure on I. When the speed measure m is a positive Radon measure on I, ρ is called a one dimensional diffusion process (ODDP) on I. We set

$$J_{\mu,\nu}(l_i) = \int_{(c,l_i)} d\mu(x) \int_{(c,x]} d\nu(y),$$

for Borel measures μ and ν on I, where c is an arbitrarily fixed point of I. Following [3], we call the end point l_i to be

$$\begin{array}{llll} (s,m,k)\text{-regular} & \text{if } J_{s,m+k}(l_i) < \infty & \text{and} & J_{m+k,s}(l_i) < \infty, \\ (s,m,k)\text{-exit} & \text{if } J_{s,m+k}(l_i) < \infty & \text{and} & J_{m+k,s}(l_i) = \infty, \\ (s,m,k)\text{-entrance} & \text{if } J_{s,m+k}(l_i) = \infty & \text{and} & J_{m+k,s}(l_i) < \infty, \\ (s,m,k)\text{-natural} & \text{if } J_{s,m+k}(l_i) = \infty & \text{and} & J_{m+k,s}(l_i) = \infty. \end{array}$$

When l_i is (s,m,k)-regular, we pose the *reflecting* or the *absorbing* boundary condition at l_i $(i = 1,2)$.

In the following we assume that supp$[m]$, the support of m, coincides with I. Hence ρ is an ODDP on I.

We consider the following symmetric bilinear form $(\mathcal{E}^\rho, \mathcal{F}^\rho)$.

$$\mathcal{E}^\rho(u,v) = \int_I \frac{du}{ds}\frac{dv}{ds}\, ds + \int_I uv\, dk,$$
$$\mathcal{F}^\rho = \{u \in L^2(I^*,m) \cap L^2(I,k) : u \text{ satisfies (2.1), (2.2) and (2.3)}\},$$

where I^* is the interval extended by adding l_i to I if l_i is (s,m,k)-regular with reflecting boundary condition, and we set $m(\{l_i\}) = 0$.

(2.1) $u(x)$ is absolutely continuous with respect to $s(x)$ on I,

(2.2) $$\frac{du}{ds} \in L^2(I,s), \qquad u(b) - u(a) = \int_{(a,b)} \frac{du}{ds}\, ds \quad (a, b \in I),$$

(2.3) $\lim_{x \to l_i} u(x) = 0$ if $|s(l_i)| < \infty$ but l_i is not (s,m,k)-regular with reflecting boundary condition $(i = 1,2)$.

Then $(\mathcal{E}^\rho, \mathcal{F}^\rho)$ is a regular, strongly local, irreducible Dirichlet form on $L^2(I^*,m)$ corresponding to the ODDP $\rho = [\rho_t,\ P^\rho_r, \zeta^\rho]$ (see [5]). Obviously the set $\{u(s(x)) : u \in C^1_c(J^*)\}$ is a core of $(\mathcal{E}^\rho, \mathcal{F}^\rho)$, where $J^* = s(I^*)$ and $C^1_c(A)$ is the set of all C^1-functions on A with compact support.

We denote by p^ρ_t the semigroup of the ODDP ρ, that is,

$$p^\rho_t f(r) = E^{P^\rho_r}[f(\rho_t)] = \int_I p^\rho(t,r,\xi) f(\xi)\, dm(\xi), \qquad t > 0,\ r \in I^*,$$

for $f \in C_b(I^*)$, where $C_b(A)$ is the set of all bounded continuous functions on a set A, E^P stands for the expectation with respect to the probability measure P, and $p^\rho(t,r,\xi)$ denotes the transition probability density of ρ with respect to dm. When l_i is (s,m,k)-regular with reflecting boundary condition, we can consider the sample paths starting at l_i. Therefore we write the ODDP ρ on I^* instead of I.

Let $\Theta = [\Theta_t, P^\Theta_\theta]$ be the spherical Brownian motion on $S^{d-1} \subset \mathbb{R}^d$ with generator $\frac{1}{2}\Delta$, Δ being the spherical Laplacian on S^{d-1}. The corresponding Dirichlet form $(\mathcal{E}^\Theta, \mathcal{F}^\Theta)$

on $L^2(S^{d-1}, m^\Theta)$ is obtained by Fukushima and Oshima [6], where m^Θ is the spherical measure on S^{d-1} (see also [13]).

We denote by p_t^Θ the semigroup of the spherical Brownian motion Θ, that is,

$$p_t^\Theta f(\theta) = E^{P_\theta^\Theta}[f(\Theta_t)] = \int_{S^{d-1}} p^\Theta(t,\theta,\varphi) f(\varphi)\, dm^\Theta(\varphi),\ t>0,\ \theta \in S^{d-1},$$

for $f \in C_b(S^{d-1})$, where $p^\Theta(t,\theta,\varphi)$ stands for the transition probability density of Θ. It is known that $p^\Theta(t,\theta,\varphi)$ is represented by spherical harmonics S_n^l, that is,

$$p^\Theta(t,\theta,\varphi) = \sum_{n=0}^{\infty} e^{-\gamma_n t} \sum_{l=1}^{\kappa(n)} S_n^l(\theta) S_n^l(\varphi), \tag{2.4}$$

where $\gamma_n = \frac{1}{2} n(n+d-2)$, $\kappa(n) = (2n+d-2)\cdot(n+d-3)!/n!\,(d-2)!$ which is the number of spherical harmonics of weight n, $\frac{1}{2}\Delta S_n^l = -\gamma_n S_n^l$, and

$$\int_{S^{d-1}} S_n^l S_n^k\, dm^\Theta = \begin{cases} 1, & l = k, \\ 0, & l \neq k, \end{cases}$$

(see [2], [8]). We set $A_{d-1} = \int_{S^{d-1}} dm^\Theta$ (the total area of the spherical surface S^{d-1}), so that $S_0^1 = A_{d-1}^{-1/2}$. Note that $\kappa(0) = 1$.

Let $\mathrm{R} = [R_t, P_r^{\mathrm{R}}, \zeta^{\mathrm{R}}]$ be the ODDP on I^* with scale function s^{R}, speed measure m^{R}, and no killing measure, where I^* is the interval extended by adding l_i to I if l_i is $(s^{\mathrm{R}}, m^{\mathrm{R}}, 0)$-regular with reflecting boundary condition, and we set $m^{\mathrm{R}}(\{l_i\}) = 0$. We turn to a skew product of $\mathrm{R} = [R_t, P_r^{\mathrm{R}}, \zeta^{\mathrm{R}}]$ and $\Theta = [\Theta_t, P_\theta^\Theta]$. It is known that the ODDP R has the local time $l^{\mathrm{R}}(t,r)$ which is continuous with respect to $(t,r) \in [0,\infty) \times I$ and satisfies $\int_0^t 1_A(R_u)\, du = \int_A l^{\mathrm{R}}(t,r)\, dm^{\mathrm{R}}(r)$, $t>0$, for every measurable set $A \subset I$ (see [8]), where 1_A is the indicator for a set A. Let ν be a Radon measure on I and assume that $\mathrm{supp}[\nu]$, the support of ν, coincides with I. Furthermore we assume that $|\int_{(l_i,c)} d\nu| < \infty$ whenever l_i is $(s^{\mathrm{R}}, m^{\mathrm{R}}, 0)$-regular (see Remark 3.3 for the reason why $|\int_{(l_i,c)} d\nu| < \infty$ is assumed). We set

$$\mathbf{f}(t) = \begin{cases} \int_I l^{\mathrm{R}}(t,r)\, d\nu(r), & t < \zeta^{\mathrm{R}}, \\ \infty, & t \geq \zeta^{\mathrm{R}}. \end{cases} \tag{2.5}$$

Since $\mathrm{supp}[\nu] = I$, we see that

$$P_r^{\mathrm{R}}(\mathbf{f}(t) > 0,\ t>0) = 1, \quad r \in I^*.$$

Let $\Xi = [\Xi_t = (R_t, \Theta_{\mathbf{f}(t)}), P_{(r,\theta)}^\Xi = P_r^{\mathrm{R}} \otimes P_\theta^\Theta,\ (r,\theta) \in I^* \times S^{d-1}, \zeta^\Xi]$ be the skew product of the ODDP R and the spherical Brownian motion Θ with respect to the PCAF $\mathbf{f}(t)$, and set

$$\mathcal{E}^\Xi(f,g) = \int_{S^{d-1}} \mathcal{E}^{\mathrm{R}}(f(\cdot,\theta), g(\cdot,\theta))\, dm^\Theta(\theta) + \int_I \mathcal{E}^\Theta(f(r,\cdot), g(r,\cdot))\, d\nu(r), \tag{2.6}$$

for $f,g \in \mathcal{C}^\Xi$, where $\mathcal{C}^\Xi$ is the set of all linear combinations of $u^{(1)}(s^{\mathrm{R}}(r))u^{(2)}(\theta)$ with $u^{(1)} \in C_c^1(J^*)$ and $u^{(2)} \in C^\infty(S^{d-1})$, where $J^* = s^{\mathrm{R}}(I^*)$. Then we have the following result.

Proposition 2.1 *The form $(\mathcal{E}^\Xi, \mathcal{C}^\Xi)$ is closable on $L^2(I^* \times S^{d-1}, m^{\mathrm{R}} \otimes m^\Theta)$. The closure $(\mathcal{E}^\Xi, \mathcal{F}^\Xi)$ is a regular Dirichlet form and it is corresponding to the skew product Ξ.*

Proof. When both of $l_i,\ i=1,2$, are $(s^{\mathrm{R}}, m^{\mathrm{R}}, 0)$-entrance, or -natural, the statement is obtained in Proposition 2.1 of [13]. In the following we do not assume that both of $l_i,\ i=1,2$, are $(s^{\mathrm{R}}, m^{\mathrm{R}}, 0)$-entrance, or -natural. Hence R is not necessarily conservative. Okura [12] considered the skew product of symmetric Markov processes which are not necessarily conservative, and we can employ his results.

Let $\widetilde{\mathrm{R}}$ be the ODDP on I with scale function s^{R}, speed measure m^{R}, and killing measure ν. Since $|\int_{(l_i,c)} d\nu| < \infty$ whenever l_i is $(s^{\mathrm{R}}, m^{\mathrm{R}}, 0)$-regular, l_i is $(s^{\mathrm{R}}, m^{\mathrm{R}}, 0)$-regular if and only if l_i is $(s^{\mathrm{R}}, m^{\mathrm{R}}, \nu)$-regular. When l_i is $(s^{\mathrm{R}}, m^{\mathrm{R}}, \nu)$-regular, we set the reflecting [resp. absorbing] boundary condition at l_i for $\widetilde{\mathrm{R}}$ according to l_i being $(s^{\mathrm{R}}, m^{\mathrm{R}}, 0)$-reflecting [resp. absorbing] for R. We extend the interval I to $\widetilde{I}$ by adding l_i to I if l_i is $(s^{\mathrm{R}}, m^{\mathrm{R}}, \nu)$-regular with reflecting boundary condition, and we set $m^{\mathrm{R}}(\{l_i\}) = 0$. We set

$$\widetilde{\mathcal{E}}(u,v) = \int_I \frac{du}{ds^{\mathrm{R}}} \frac{dv}{ds^{\mathrm{R}}}\, ds^{\mathrm{R}} + \int_I uv\, d\nu,$$
$$\widetilde{\mathcal{F}} = \{u \in L^2(\tilde{I}, m^{\mathrm{R}}) \cap L^2(I,\nu) : u \text{ satisfies (2.1), (2.2) and (2.3) with } s^{\mathrm{R}} \text{ and } \nu \text{ in place of } s \text{ and } k, \text{ respectively}\}.$$

By means of Theorem 2.2 of [5], $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ is a regular, strongly local, irreducible Dirichlet form on $L^2(\widetilde{I}, m^{\mathrm{R}})$ corresponding to the ODDP $\widetilde{\mathrm{R}}$. We note $\widetilde{I} = I^*$. Therefore the set $\{u^{(1)}(s^{\mathrm{R}}(r)) : u^{(1)} \in C_c^1(J^*)\}$ is a core of $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$. Thus by means of Theorem 3.1 of [12] we obtain the proposition. □

3 Feller property of the skew product

Let $\Xi = [\Xi_t = (R_t, \Theta_{\mathbf{f}(t)}), P^{\Xi}_{(r,\theta)} = P^{\mathrm{R}}_r \otimes P^{\Theta}_\theta,\ (r,\theta) \in I^* \times S^{d-1}, \zeta^{\Xi}]$ be the skew product of the ODDP R and the spherical Brownian motion Θ with respect to the PCAF $\mathbf{f}(t)$ defined in the preceding section. We show Feller property of the skew product Ξ. Denote by p^{Ξ}_t the semigroup of the skew product Ξ, that is,

$$p^{\Xi}_t f(r,\theta) = E^{P^{\mathrm{R}}_r \otimes P^{\Theta}_\theta}[f(R_t, \Theta_{\mathbf{f}(t)})], \qquad t>0,\ (r,\theta) \in I^* \times S^{d-1},$$

for $f \in C_b(I^* \times S^{d-1})$. By virtue of (2.4) we obtain the following

$$\begin{aligned} p^{\Xi}_t f(r,\theta) &= \int_{S^{d-1}} E^{P^{\mathrm{R}}_r}\left[f(R_t,\varphi)\, p^{\Theta}(\mathbf{f}(t),\theta,\varphi)\right] dm^{\Theta}(\varphi) \\ &= \sum_{n=0}^{\infty} \sum_{l=1}^{\kappa(n)} S^l_n(\theta) \int_{S^{d-1}} S^l_n(\varphi) E^{P^{\mathrm{R}}_r}\left[f(R_t,\varphi) e^{-\gamma_n \mathbf{f}(t)}\right] dm^{\Theta}(\varphi). \end{aligned} \tag{3.1}$$

Since $E^{P^{\mathrm{R}}_r}\left[f(R_t,\eta)e^{-\gamma_n \mathbf{f}(t)}\right]$ is continuous in $r \in I^*$ (see [8]), we immediately obtain the following result by means of (3.1), so we omit the proof.

Proposition 3.1 *Let $f \in C_b(I^* \times S^{d-1})$ and $t > 0$. Then $p^{\Xi}_t f \in C_b(I^* \times S^{d-1})$.*

Let us observe the behavior of $p^{\Xi}_t f(r,\theta)$ as $r \to l_i$.

Theorem 3.2 *Let $i = 1,2,\ t>0,\ \theta \in S^{d-1}$ and $f \in C_b(I^* \times S^{d-1})$.*

(i) *If l_i is $(s^{\mathrm{R}}, m^{\mathrm{R}}, 0)$-regular with reflecting boundary condition, then*

$$\lim_{r\to l_i} p_t^{\Xi} f(r,\theta) = p_t^{\Xi} f(l_i,\theta).$$

(ii) *If the end point l_i is $(s^{\mathrm{R}}, m^{\mathrm{R}}, 0)$-regular with absorbing boundary condition or -exit, then*

$$\lim_{r\to l_i} p_t^{\Xi} f(r,\theta) = 0. \tag{3.2}$$

(iii) *If the end point l_i is $(s^{\mathrm{R}}, m^{\mathrm{R}}, 0)$-entrance, then there exist the limits* $\lim_{r\to l_i} E^{P_r^{\mathrm{R}}}\left[f(R_t,\theta)e^{-C\mathbf{f}(t)}\right]$ *and* $\lim_{r\to l_i} p_t^{\Xi} f(r,\theta)$ *for each* $C \geq 0$. *In particular, if the measure ν satisfies*

$$\left|\int_{(c,l_i)} s^{\mathrm{R}}(r)\, d\nu(r)\right| = \infty, \tag{3.3}$$

then

$$\lim_{r\to l_i} E^{P_r^{\mathrm{R}}}\left[f(R_t,\theta)\, e^{-C\mathbf{f}(t)}\right] = 0, \quad C > 0. \tag{3.4}$$

$$\lim_{r\to l_i} p_t^{\Xi} f(r,\theta) = \frac{1}{A_{d-1}} \int_{S^{d-1}} \lim_{r\to l_i} E^{P_r^{\mathrm{R}}}\left[f(R_t,\varphi)\right] dm^{\Theta}(\varphi). \tag{3.5}$$

Note that the limit (3.5) *is independent of θ.*
(iv) *If the end point l_i is $(s^{\mathrm{R}}, m^{\mathrm{R}}, 0)$-natural and there exists the limit* $\lim_{r\to l_i} f(r,\theta) = 0$ *for each $\theta \in S^{d-1}$, then* (3.2) *holds true.*

Proof. (i) Assume that the end point l_i is $(s^{\mathrm{R}}, m^{\mathrm{R}}, 0)$-regular with reflecting boundary condition. The statement is obvious by Proposition 3.1.

(ii) Assume that the end point l_i is $(s^{\mathrm{R}}, m^{\mathrm{R}}, 0)$-regular with absorbing boundary condition or -exit. Then, by virtue of properties of ODDP R,

$$\lim_{r\to l_i} \left|E^{P_r^{\mathrm{R}}}\left[f(R_t,\varphi)e^{-C\mathbf{f}(t)}\right]\right| \leq \lim_{r\to l_i} E^{P_r^{\mathrm{R}}}\left[|f(R_t,\varphi)|\right] = 0, \quad C \geq 0,\ \varphi \in S^{d-1}. \tag{3.6}$$

Combining this with (3.1), we obtain (3.2).

(iii) Assume that the end point l_i is $(s^{\mathrm{R}}, m^{\mathrm{R}}, 0)$-entrance. It follows from properties of ODDP R and (3.1) that there exist the limits $\lim_{r\to l_i} E^{P_r^{\mathrm{R}}}\left[f(R_t,\theta)e^{-C\mathbf{f}(t)}\right]$ and $\lim_{r\to l_i} p_t^{\Xi} f(r,\theta)$ for each $C \geq 0$ and $\theta \in S^{d-1}$.

We next show (3.4) and (3.5) under the assumption (3.3). We may set $i = 1$. In the case that l_2 is $(s^{\mathrm{R}}, m^{\mathrm{R}}, 0)$-entrance or -natural, (3.4) and (3.5) are obtained in Theorem 3.2 of [13]. The proof of [13] is available in this paper, too. So we only give the outline of the proof. We note

$$\zeta^{\mathrm{R}} = \begin{cases} \sigma_{l_2}^{\mathrm{R}}, & \text{if } l_2 \text{ is } (s^{\mathrm{R}}, m^{\mathrm{R}}, 0)\text{-regular with absorbing boundary condition, or -exit,} \\ \infty, & \text{otherwise,} \end{cases}$$

where σ_a^{R} is the first hitting time to a of the ODDP R. Let $C > 0$ and $t > 0$. Then we see the following.

$$\begin{aligned} E^{P_r^{\mathrm{R}}}\left[e^{-C\mathbf{f}(t)}\right] &= E^{P_r^{\mathrm{R}}}\left[e^{-C\mathbf{f}(t)},\ t < \zeta^{\mathrm{R}}\right] \\ &= E^{P_r^{\mathrm{R}}}\left[e^{-C\mathbf{f}(t)},\ t < \zeta^{\mathrm{R}},\ t < \sigma_a^{\mathrm{R}}\right] + E^{P_r^{\mathrm{R}}}\left[e^{-C\mathbf{f}(t)},\ t < \zeta^{\mathrm{R}},\ t \geq \sigma_a^{\mathrm{R}}\right] \end{aligned} \tag{3.7}$$

$$\le P_r^{\mathrm{R}}\left(t < \sigma_a^{\mathrm{R}}\right) + E^{P_r^{\mathrm{R}}}\left[e^{-C\mathbf{f}(\sigma_a^{\mathrm{R}})}\right],$$

for any $a \in I$. Since l_1 is $(s^{\mathrm{R}}, m^{\mathrm{R}}, 0)$-entrance, we get

$$\limsup_{a\to l_1}\limsup_{r\to l_1} P_r^{\mathrm{R}}\left(t < \sigma_a^{\mathrm{R}}\right) = 0. \tag{3.8}$$

Let Q be the ODDP on I with scale function s^{R}, speed measure ν, and no killing measure. Note that l_1 is $(s^{\mathrm{R}}, \nu, 0)$-natural by virtue of (3.3), and the law of $\left[\mathbf{f}(\sigma_a^{\mathrm{R}}), P_r^{\mathrm{R}}, r \in (l_1, a)\right]$ is the same as that of $\left[\sigma_a^{\mathrm{Q}}, P_r^{\mathrm{Q}}, r \in (l_1, a)\right]$. Therefore, by using properties of Q,

$$\lim_{r\to l_1} E^{P_r^{\mathrm{R}}}\left[e^{-C\mathbf{f}(\sigma_a^{\mathrm{R}})}\right] = \lim_{r\to l_1} E^{P_r^{\mathrm{Q}}}\left[e^{-C\sigma_a^{\mathrm{Q}}}\right] = 0. \tag{3.9}$$

Combining this with (3.7) and (3.8), we have

$$\lim_{r\to l_1} E^{P_r^{\mathrm{R}}}\left[e^{-C\mathbf{f}(t)}\right] = 0,$$

from which (3.4) follows. (3.5) is obtained by (3.1) and (3.4).

(iv) Assume that the end point l_i is $(s^{\mathrm{R}}, m^{\mathrm{R}}, 0)$-natural and there exists the limit $\lim_{r\to l_i} f(r,\theta) = 0$ for each $\theta \in S^{d-1}$. Then, by virtue of properties of ODDP R, we get (3.6). Therefore (3.2) follows from (3.1) and (3.6). □

Remark 3.3 (a) Let $l_1 = 0$ be $(s^{\mathrm{R}}, m^{\mathrm{R}}, 0)$-entrance and assume (3.3) for $i = 1$. Then the statement (ii) implies that the skew product Ξ on $I^* \times S^{d-1}$ is uniquely extended to the diffusion process on $I^* \times S^{d-1} \cup \{\mathbf{0}\}$, where $\mathbf{0}$ is the original point of $\mathbb{R}^d$.

(b) When we consider the skew product Ξ of the ODDP R on I^* and the spherical Brownian motion Θ with respect to PCAF $\mathbf{f}(t)$ of R following [6], [11], and [12], the Revuz measure ν of $\mathbf{f}(t)$ is a Radon measure on I^*. Therefore we assume $|\int_{(l_i,c)} d\nu| < \infty$ whenever l_i is $(s^{\mathrm{R}}, m^{\mathrm{R}}, 0)$-regular with reflecting boundary condition. Here we show that the condition $|\int_{(l_i,c)} d\nu| < \infty$ is necessary for the Revuz measure of PCAF $\mathbf{f}(t)$ of the skew product Ξ if l_i is $(s^{\mathrm{R}}, m^{\mathrm{R}}, 0)$-regular with reflecting boundary condition.

Assume that the end point l_1 is $(s^{\mathrm{R}}, m^{\mathrm{R}}, 0)$-regular with reflecting boundary condition, and the measure ν satisfies

$$\left|\int_{(l_1,c)} d\nu(r)\right| = \infty. \tag{3.10}$$

Let $f \in C_b(I^* \times S^{d-1})$. Then, by means of properties of ODDPs, we see the following.

$$E^{P_{l_1}^{\mathrm{R}}}[f(R_t,\theta)] = \lim_{r\to l_1} E^{P_r^{\mathrm{R}}}[f(R_t,\theta)], \quad t > 0,\ \theta \in S^{d-1}.$$

(3.7) holds true in this case, too. Since l_1 is $(s^{\mathrm{R}}, m^{\mathrm{R}}, 0)$-regular with reflecting boundary condition, we get (3.8). Note that l_1 is $(s^{\mathrm{R}}, \nu, 0)$-exit or -natural by virtue of (3.10), and $\mathbf{f}(\sigma_a^{\mathrm{R}})$ is the first passage time to a of the ODDP Q with scale function s^{R}, speed measure ν, and no killing measure. By means of some properties of ODDP Q, we get (3.9) and hence (3.4). Combining (3.4) with (3.1), we get (3.5). Let $\widetilde{\mathrm{R}}$ and $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ be the ODDP and the corresponding Dirichlet form given in the proof of Proposition 2.1, respectively. Since $|s^{\mathrm{R}}(l_1)| < \infty$ and l_1 is $(s^{\mathrm{R}}, m^{\mathrm{R}}, \nu)$-exit or -natural by virtue of (3.10), we find by (2.3) that

$$\lim_{r\to l_1} u(r) = 0 \quad \text{for } u \in \widetilde{\mathcal{F}}.$$

Applying Okura's result [12], we get that all functions belonging to the core $\mathcal{C}^{\Xi}$ vanish on $\{l_1\} \times S^{d-1}$, where $\mathcal{C}^{\Xi}$ is given right after (2.6). By virtue of general theory on Dirichlet form (cf. [7]),

$$p_t^{\Xi} f \in \mathcal{F}^{\Xi} \quad \text{for } f \in L^2(I^* \times S^{d-1}, m^{\mathrm{R}} \otimes m^{\Theta}).$$

Therefore

$$\lim_{r \to l_1} p_t^{\Xi} f(r,\theta) = \frac{1}{A_{d-1}} \int_{S^{d-1}} E^{P_{l_1}^{\mathrm{R}}}[f(R_t, \varphi)]\, dm^{\Theta}(\varphi) = 0,\ \theta \in S^{d-1},$$

for $f \in C_b(I^* \times S^{d-1}) \cap L^2(I^* \times S^{d-1}, m^{\mathrm{R}} \otimes m^{\Theta})$, from which

$$E^{P_{l_1}^{\mathrm{R}}}[u(R_t)] = 0, \quad \text{for } u \in C_b(I^*) \cap L^2(I^*, m^{\mathrm{R}}).$$

This contradicts the fact that R is the ODDP and l_1 is regular with reflecting boundary condition. Thus the measure ν must satisfy $|\int_{(l_1,c)} d\nu| < \infty$ if l_1 is $(s^{\mathrm{R}}, m^{\mathrm{R}}, 0)$-regular with reflecting boundary condition. In the case l_1 being $(s^{\mathrm{R}}, m^{\mathrm{R}}, 0)$-regular with absorbing boundary condition, the corresponding ODDP R^0 is the part on I of R. Therefore we assume $|\int_{(l_1,c)} d\nu| < \infty$ in this case, too.

4 Feller property of time changed processes

Let $\Xi = [\Xi_t = (R_t, \Theta_{\mathbf{f}(t)}), P^{\Xi}_{(r,\theta)} = P_r^{\mathrm{R}} \otimes P_\theta^{\Theta},\ (r,\theta) \in I^* \times S^{d-1},\ \zeta^{\Xi}]$ be the skew product of the ODDP R and the spherical Brownian motion Θ with respect to the PCAF $\mathbf{f}(t)$ defined in Section 2. In this section we consider a time changed process of Ξ and show its Feller property.

Let Λ be a closed set of I^* and μ be the restriction of m^{R} to Λ. We set

$$\mathbf{g}(t) = \begin{cases} \displaystyle\int_{I^*} l^{\mathrm{R}}(t,r)\, d\mu(r) = \int_{\Lambda} l^{\mathrm{R}}(t,r)\, dm^{\mathrm{R}}(r), & 0 \le t < \zeta^{\mathrm{R}}, \\ \infty, & t \ge \zeta^{\mathrm{R}}. \end{cases} \tag{4.1}$$

We denote by $\tau(t)$ the right continuous inverse of $\mathbf{g}(t)$. We consider the time changed process $\mathrm{X} = [X_t = (R_{\tau(t)}, \Theta_{\mathbf{f}(\tau(t))}), P^{\mathrm{X}}_{(r,\theta)} = P_r^{\mathrm{R}} \otimes P_\theta^{\Theta},\ (r,\theta) \in I^* \times S^{d-1},\ \zeta^{\mathrm{X}}]$. Note that $\zeta^{\mathrm{X}} = \mathbf{g}(\zeta^{\mathrm{R}}-)$. Denote by p_t^{X} the semigroup of X, that is,

$$p_t^{\mathrm{X}} f(r,\theta) = E^{P_r^{\mathrm{R}} \otimes P_\theta^{\Theta}}[f(R_{\tau(t)}, \Theta_{\mathbf{f}(\tau(t))})], \qquad t > 0,\ (r,\theta) \in I^* \times S^{d-1},$$

for $f \in C_b(I^* \times S^{d-1})$. By virtue of (2.4) we obtain the following

$$\begin{aligned} &p_t^{\mathrm{X}} f(r,\theta) \\ &= \int_{S^{d-1}} E^{P_r^{\mathrm{R}}} \left[f(R_{\tau(t)}, \varphi)\, p^{\Theta}(\mathbf{f}(\tau(t)), \theta, \varphi) \right] dm^{\Theta}(\varphi) \\ &= \sum_{n=0}^{\infty} \sum_{l=1}^{\kappa(n)} S_n^l(\theta) \int_{S^{d-1}} S_n^l(\varphi) E^{P_r^{\mathrm{R}}} \left[f(R_{\tau(t)}, \varphi) e^{-\gamma_n \mathbf{f}(\tau(t))} \right] dm^{\Theta}(\varphi). \end{aligned} \tag{4.2}$$

Note that the time changed process $\mathrm{U} = [R_{\tau(t)}, P_r^{\mathrm{R}},\ r \in I^*,\ \zeta^{\mathrm{U}}]$ is the ODGDP on I^* with scale function s^{R}, speed measure μ and no killing measure, where $\zeta^{\mathrm{U}} = \mathbf{g}(\zeta^{\mathrm{R}}-)$. We set $\Gamma = \Lambda \times S^{d-1}$. The time changed process X is essentially defined on Γ. Since $E^{P_r^{\mathrm{R}}} \left[f(R_{\tau(t)}, \varphi) e^{-\gamma_n \mathbf{f}(\tau(t))} \right]$ is continuous in $r \in \Lambda$ (see [8], [9], [14]), the following result is obvious by means of (4.2). So we omit the proof.

Proposition 4.1 *Let $f \in C_b(\Gamma)$ and $t > 0$. Then $p_t^{\mathrm{X}} f \in C_b(\Gamma)$.*

We observe the behavior of $p_t^{\mathrm{X}} f(r,\theta)$ as $r\,(\in \Lambda) \to l_1$ [resp. l_2] when $l_1 = \inf \Lambda$ [resp. $l_2 = \sup \Lambda$]. In the same way as in Theorem 4.2 of [13], the following result follows from Theorem 3.2. So we omit the proof.

Theorem 4.2 *Let $f \in C_b(\Gamma)$, $t > 0$ and $\theta \in S^{d-1}$. The following properties hold true for the end point l_i satisfying $l_1 = \inf \Lambda$ or $l_2 = \sup \Lambda$.*
(i) *If l_i is $(s^{\mathrm{R}}, \mu, 0)$-regular with reflecting boundary condition, then*

$$\lim_{r(\in\Lambda)\to l_i} p_t^{\mathrm{X}} f(r,\theta) = p_t^{\mathrm{X}} f(l_i,\theta).$$

(ii) *If the end point l_i is $(s^{\mathrm{R}}, \mu, 0)$-regular with absorbing boundary condition or -exit, then*

$$\lim_{r(\in\Lambda)\to l_i} p_t^{\mathrm{X}} f(r,\theta) = 0. \tag{4.3}$$

(iii) *If the end point l_i is $(s^{\mathrm{R}}, \mu, 0)$-entrance, then there exist the limits $\lim_{r(\in\Lambda)\to l_i} E^{P_r^{\mathrm{R}}}\left[f(R_{\tau(t)},\theta)e^{-C\mathbf{f}(\tau(t))}\right]$ and $\lim_{r(\in\Lambda)\to l_i} p_t^{\mathrm{X}} f(r,\theta)$ for each $C \geq 0$. In particular, if the measure ν satisfies (3.3), then*

$$\lim_{r(\in\Lambda)\to l_i} E^{P_r^{\mathrm{R}}}\left[f(R_{\tau(t)},\theta)\, e^{-C\mathbf{f}(\tau(t))}\right] = 0, \quad C > 0.$$

$$\lim_{r(\in\Lambda)\to l_i} p_t^{\mathrm{X}} f(r,\theta) = \frac{1}{A_{d-1}} \int_{S^{d-1}} \lim_{r(\in\Lambda)\to l_i} E^{P_r^{\mathrm{R}}}\left[f(R_{\tau(t)},\varphi)\right] dm^{\Theta}(\varphi),$$

which is independent of θ.
(iv) *If the end point l_i is $(s^{\mathrm{R}}, \mu, 0)$-natural and there exists the limit $\lim_{r(\in\Lambda)\to l_i} f(r,\theta) = 0$ for each $\theta \in S^{d-1}$, then (4.3) holds true.*

5 Dirichlet form of the time changed process

In this section, we derive the Dirichlet form $(\mathcal{E}^{\mathrm{X}}, \mathcal{F}^{\mathrm{X}})$ of the time changed process X defined in the preceding section. X is a time changed process of Ξ. Ξ is the skew product of R and Θ with respect to $\mathbf{f}$ defined by (2.5), and the Dirichlet form $(\mathcal{E}^{\Xi}, \mathcal{F}^{\Xi})$ corresponding to Ξ is given in Proposition 2.1. Since μ is the restriction of m^{R} to Λ, the measure $\mu \otimes m^{\Theta}$ charges no set of zero $\mathcal{E}^{\Xi}$-capacity. We note that $\mathbf{g}(t)$ defined by (4.1) is a PCAF of Ξ and $P^{\Xi}_{(r,\theta)}(\mathbf{g}(t) > 0,\ t > 0) = 1$ for $(r,\theta) \in \Gamma$. Since $(\mathcal{E}^{\Xi}, \mathcal{F}^{\Xi})$ is a regular Dirichlet form on $L^2(I^* \times S^{d-1}, m^{\mathrm{R}} \otimes m^{\Theta})$, $\mathcal{C} := \mathcal{F}^{\Xi} \cap C_c(I^* \times S^{d-1})$ is a special standard core of $(\mathcal{E}^{\Xi}, \mathcal{F}^{\Xi})$ (see [7]). Hence, employing Theorem 6.2.1 in [7], we see that the Dirichlet form $(\mathcal{E}^{\mathrm{X}}, \mathcal{F}^{\mathrm{X}})$ is regular on $L^2(\Gamma, \mu \otimes m^{\Theta})$ and has $\mathcal{C}|_{\Gamma}$ as a core, where $\mathcal{C}|_{\Gamma} = \{u|_{\Gamma} : u \in \mathcal{C}\}$. Obviously $\mathcal{C}^{\Xi}|_{\Gamma} \subset \mathcal{C}|_{\Gamma}$. If $\int_{\Lambda} ds^{\mathrm{R}} > 0$, then there exists the limit

$$\partial^*_{s^{\mathrm{R}}} f(r,\theta) = \lim_{r'(\in\Lambda)\to r} \frac{f(r',\theta) - f(r,\theta)}{s^{\mathrm{R}}(r') - s^{\mathrm{R}}(r)} = \lim_{r'\to r} \frac{u(r',\theta) - u(r,\theta)}{s^{\mathrm{R}}(r') - s^{\mathrm{R}}(r)},$$

for ds^{R}-a.e. $r \in \Lambda$ and every $\theta \in S^{d-1}$, $u \in \mathcal{C}^{\Xi}$ and $f = u|_{\Gamma}$ (see Lemma 5.1 of [13]). Assume that $I \setminus \Lambda \neq \emptyset$. We write that $I \setminus \Lambda = \cup_{k\in K} I_k$, a finite or a countable disjoint union of open intervals $I_k = (a_k, b_k)$ with the end points belonging to $\Lambda \cup \{l_1, l_2\}$. Let $J_k^{i,j}(\theta,\varphi)$, $i,j = 1,2$, be the jump measure densities defined by (5.13)–(5.16) of [13], that is,

$$J_k^{1,1}(\theta,\varphi) = \lim_{r\downarrow a_k} D_{s^{\mathrm{R}}(r)} G_{k,2}(r;\theta,\varphi).$$

$$J_k^{1,2}(\theta,\varphi) = \lim_{r\downarrow a_k} D_{s^{\mathrm{R}}(r)} G_{k,1}(r;\theta,\varphi).$$
$$J_k^{2,1}(\theta,\varphi) = -\lim_{r\uparrow b_k} D_{s^{\mathrm{R}}(r)} G_{k,2}(r;\theta,\varphi).$$
$$J_k^{2,2}(\theta,\varphi) = -\lim_{r\uparrow b_k} D_{s^{\mathrm{R}}(r)} G_{k,1}(r;\theta,\varphi),$$

for $r \in I_k = (a_k, b_k)$ and $\theta,\ \varphi \in S^{d-1}$, where

$$G_{k,1}(r;\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{l=1}^{\kappa(n)} S_n^l(\theta) S_n^l(\varphi) E^{P_r^{\mathrm{R}}}\left[e^{-\gamma_n \mathbf{f}(\sigma_{b_k}^{\mathrm{R}})}; \sigma_{b_k}^{\mathrm{R}} < \sigma_{a_k}^{\mathrm{R}},\ \sigma_{b_k}^{\mathrm{R}} < \zeta^{\mathrm{R}}\right],$$
$$G_{k,2}(r;\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{l=1}^{\kappa(n)} S_n^l(\theta) S_n^l(\varphi) E^{P_r^{\mathrm{R}}}\left[e^{-\gamma_n \mathbf{f}(\sigma_{a_k}^{\mathrm{R}})}; \sigma_{a_k}^{\mathrm{R}} < \sigma_{b_k}^{\mathrm{R}},\ \sigma_{a_k}^{\mathrm{R}} < \zeta^{\mathrm{R}}\right].$$

Let $\mathcal{M}$ be the product measure $m^{\Theta} \otimes m^{\Theta}$. Then the Dirichlet form $(\mathcal{E}^{\mathrm{X}}, \mathcal{F}^{\mathrm{X}})$ is given by the following.

Theorem 5.1 *Assume $I \setminus \Lambda \neq \emptyset$ and let $f \in \mathcal{C}^{\Xi}|_{\Gamma}$. Then*

$$\begin{aligned}
\mathcal{E}^{\mathrm{X}}(f,f) &= \int_{\Gamma} \partial_{s^{\mathrm{R}}}^{*} f(r,\theta)^2\, ds^{\mathrm{R}}(r)\, dm^{\Theta}(\theta) + \int_{\Lambda} \mathcal{E}^{\Theta}(f(r,\cdot), f(r,\cdot))\, d\nu(r) \\
&+ \frac{1}{2}\sum_{k\in K: l_1 < a_k < b_k \le l_2} \int_{S^{d-1}\times S^{d-1}} \{f(a_k,\theta) - f(a_k,\varphi)\}^2 J_k^{1,1}(\theta,\varphi)\, d\mathcal{M}(\theta,\varphi) \\
&+ \frac{1}{2}\sum_{k\in K: l_1 \le a_k < b_k < l_2} \int_{S^{d-1}\times S^{d-1}} \{f(b_k,\theta) - f(b_k,\varphi)\}^2 J_k^{2,2}(\theta,\varphi)\, d\mathcal{M}(\theta,\varphi) \\
&+ \frac{1}{2}\sum_{k\in K: l_1 < a_k < b_k < l_2} \int_{S^{d-1}\times S^{d-1}} \{f(a_k,\theta) - f(b_k,\varphi)\}^2 J_k^{1,2}(\theta,\varphi)\, d\mathcal{M}(\theta,\varphi) \\
&+ \frac{1}{2}\sum_{k\in K: l_1 < a_k < b_k < l_2} \int_{S^{d-1}\times S^{d-1}} \{f(b_k,\theta) - f(a_k,\varphi)\}^2 J_k^{2,1}(\theta,\varphi)\, d\mathcal{M}(\theta,\varphi) \\
&+ I_1(f) + I_2(f).
\end{aligned}$$

Here the first term of the right hand side vanishes in case that $\int_{\Lambda} ds^{\mathrm{R}}(r) = 0$. The last two terms $I_i(f),\ i = 1,2$ should be read as

$$I_1(f) = \begin{cases} \dfrac{1}{s^{\mathrm{R}}(b_k) - s^{\mathrm{R}}(l_1)} \displaystyle\int_{S^{d-1}} f(b_k,\theta)^2\, dm^{\Theta}(\theta) & \\ \qquad \textit{if } l_1 = a_k < b_k < l_2,\ s^{\mathrm{R}}(l_1) > -\infty,\ \textit{but } l_1 \textit{ is not} \\ \qquad (s^{\mathrm{R}}, m^{\mathrm{R}}, 0)\textit{-regular with reflecting boundary condition,} \\ 0 \qquad \textit{otherwise,} \end{cases}$$

$$I_2(f) = \begin{cases} \dfrac{1}{s^{\mathrm{R}}(l_2) - s^{\mathrm{R}}(a_k)} \displaystyle\int_{S^{d-1}} f(a_k,\theta)^2\, dm^{\Theta}(\theta) & \\ \qquad \textit{if } l_1 < a_k < b_k = l_2,\ s^{\mathrm{R}}(l_2) < \infty,\ \textit{but } l_2 \textit{ is not} \\ \qquad (s^{\mathrm{R}}, m^{\mathrm{R}}, 0)\textit{-regular with reflecting boundary condition,} \\ 0 \qquad \textit{otherwise,} \end{cases}$$

The statement of Theorem 5.1 is same as that of Theorem 5.6 of [13] except $I_i(f),\ i=1,2$. $I_i(f)$ $(i=1,2)$ of Theorem 5.1 are obtained in the same way as Lemma 5.4 of [13]. So the proof is omitted.

References

[1] Z.-Q. Chen and M. Fukushima: *Symmetric Markov Processes, Time Change, and Boundary Theory*, London Math. Soc. Monogr. Ser, 35, Princeton University Press, 2012.

[2] A. Erdélyi: *Higher transcendental functions* **1,2**, New York 1953.

[3] W. Feller: The parabolic differential equations and the associated semi-groups of transformations, Ann. of Math.**55** (1952), 468–519.

[4] M. Fukushima, From one dimensional diffusions to symmetric Markov processes, Stochastic Process. Appl., **120** (2010), 590-604. (Special issue A tribute to Kiyosi Itô)

[5] M. Fukushima, On general boundary conditions for one-dimensional diffusions with symmetry, J. Math. Soc. Japan, **66** (2014), 289-316

[6] M. Fukushima and Y. Oshima: On the Skew Product of Symmetric Diffusion Processes, Forum Math.**1** (1989), 103–142

[7] M. Fukushima, Y. Oshima and M. Takeda: *Dirichlet Forms and Symmetric Markov Processes*, Walter de Gruyter, Berlin, 1994

[8] K. Itô and H. P. McKean, Jr.: *Diffusion Processes and their Sample Paths*, Springer-Verlag, New York, 1974.

[9] S. Kotani and S. Watanabe: Krein's spectral theory of strings and generalized diffusion processes, Functional Analysis in Markov Processes (M. Fukushima, ed.), Lecture Notes on Math.**923** Springer (1982), 235–259.

[10] H. P. McKean: Elementary solutions for certain parabolic partial differential equations, Trans. Amer. Math. Soc.**82** (1956), 519–548.

[11] H. Ôkura: Recurrence criteria for skew products of symmetric Markov processes, Forum Math.**1** (1989), 331–357.

[12] H. Ôkura: A new approach to the skew product of symmetric Markov processes, Memoirs of the Faculty of Engineering and Design, Kyoto Institute of Technology **46** (1997), 1–12.

[13] T. Takemura and M. Tomisaki: Feller property of skew product diffusion processes, Osaka J. Math. **48** (2011), 269–290.

[14] S. Watanabe: On time inversion of one-dimensional diffusion processes, Z. Wahrsch. verw. Geb.**31** (1975), 115–124.

Feller property and Dirichlet forms for skew product diffusion processes and their time change

TAKEMURA Tomoko and TOMISAKI Matsuyo

We are concerned with the skew product of a one dimensional diffusion process R on an open interval and the spherical Brownian motion. The skew product is defined by a positive continuous additive functional of R. We study Feller property of the skew product and the time change. Further we present Dirichlet forms of the skew product and the time change. In 2011, we obtained the Dirichlet form corresponding to the skew product under the assumption that both end point of the state space of R are entrance or natural in the sense of Feller. In this article we remove that assumption and tackle the same problem as the paper we wrote in 2011. Okura (1998) investigated Dirichlet forms corresponding to the skew product of Markov processes which are not necessarily conservative. He regarded the Revuz measure of positive continuous additive functional as a killing measure, and considered a new Dirichlet form with killing part. Then he derived Dirichlet forms corresponding to the skew product. Fukushima (2014) made clear Dirichlet forms of one dimensional diffusion process with killing measure. Combining their results, we can obtain the Dirichlet form corresponding to the skew product in general case. In our argument corresponding to the case that the end point of state space of R is regular with reflecting boundary condition in the sense of Feller, we assume that the Revuz measure of positive continuous additive functional is bounded near the end point. This assumption is necessary and sufficient for the Revuz measure in some sense. We show this fact following the same argument as for Feller property of the skew product.