# State of boundaries for harmonic transforms of one-dimensional generalized diffusion processes

TAKEMURA Tomoko *†

## 1 Introduction

Let $m$ be a right continuous nondecreasing function on an open interval $I = (l_1, l_2)$, where $-\infty \le l_1 < l_2 \le \infty$, $s$ be a continuous increasing function on $I$, and $k$ be a right continuous nondecreasing function on $I$. We assume that the support of the measure $dm(x)$ on $I$ induced by $m(x)$ is equal to $I$. For a function $u$ on $I$, we set $u(l_i) = \lim_{x\to l_i, x\in I} u(x)$ if there exists the limit, for $i = 1,\ 2$. We set $I^* = I \cup \{x; x = l_i \text{ with } |m(l_i)| + |s(l_i)| + |k(l_i)| < \infty,\ i = 1, 2\}$. Let us fix a point $c_o \in I$ arbitrarily and set

$$J_{\mu,\nu}(x) = \int_{(c_o,x]} d\mu(y) \int_{(c_o,y]} d\nu(z),$$

for $x \in I$, where $d\mu$ and $d\nu$ are Borel measures on $I$. and the integral $\int_{(a,b]}$ is read as $-\int_{(b,a]}$ if $a > b$. Following [1], we call the boundary $l_i$ to be

$$\begin{array}{lllll}
(s,m,k)\text{-regular} & \text{if} & J_{s,m+k}(l_i) < \infty & \text{and} & J_{m+k,s}(l_i) < \infty, \\
(s,m,k)\text{-exit} & \text{if} & J_{s,m+k}(l_i) < \infty & \text{and} & J_{m+k,s}(l_i) = \infty, \\
(s,m,k)\text{-entrance} & \text{if} & J_{s,m+k}(l_i) = \infty & \text{and} & J_{m+k,s}(l_i) < \infty, \\
(s,m,k)\text{-natural} & \text{if} & J_{s,m+k}(l_i) = \infty & \text{and} & J_{m+k,s}(l_i) = \infty.
\end{array}$$

We note that

$$\begin{array}{llll}
\text{if } l_i \text{ is } (s,m,k)\text{ -regular}, & |(m+k)(l_i)| < \infty & \text{and} & |s(l_i)| < \infty, \\
\text{if } l_i \text{ is } (s,m,k)\text{ -exit}, & |(m+k)(l_i)| = \infty & \text{and} & |s(l_i)| < \infty, \\
\text{if } l_i \text{ is } (s,m,k)\text{ -entrance}, & |(m+k)(l_i)| < \infty & \text{and} & |s(l_i)| = \infty, \\
\text{if } l_i \text{ is } (s,m,k)\text{ -natural}, & |(m+k)(l_i)| = \infty & \text{or} & |s(l_i)| = \infty.
\end{array}$$

Let $D(\mathcal{G})$ be the space of all functions $u \in L^2(I,m)$ which have continuous representatives $u$ (we use the same symbol) satisfying the following conditions:

i) There exist two constants $A,\ B$ and a function $h_u \in L^2(I,m)$ such that

$$\begin{aligned}
u(x) =& A + Bs(x) + \int_{(c_o,x]} \{s(x) - s(y)\} h_u(y)\, dm(y) \\
&+ \int_{(c_o,x]} \{s(x) - s(y)\} u(y)\, dk(y), \quad x \in I. \qquad (1.1)
\end{aligned}$$

* School of Interdisciplinary Research of Scientific Phenomena and Information

† Research Fellow of the Japan Society for the Promotion of Sciense

ii) If $l_i$ is regular, then $u(l_i) = 0$ for each $i = 1,\ 2$.

By virtue of (1.1), $h_u$ is uniquely determined as a function of $L^2(I, m)$ if it exists. The operator $\mathcal{G}$ from $D(\mathcal{G})$ into $L^2(I, m)$ is defined by $\mathcal{G}u = h_u$, and it is called the one-dimensional generalized diffusion operator with the speed measure $m$, the scale function $s$, and the killing measure $k$ (ODGDO with $(s, m, k)$ for short). In the following, for a measurable functions $u$ on $I$, $D_s u(x)$ stands for the right derivative with respect to $s(x)$, that is, $D_s u(x) = \lim_{\varepsilon \downarrow 0}\{u(x+\varepsilon) - u(x)\}/\{s(x+\varepsilon) - s(x)\}$, provided it exists. It is obvious that $u \in D(\mathcal{G})$ has the right derivative $D_s u$ and it satisfies

$$D_s u(y) - D_s u(x) = \int_{(x,y]} \mathcal{G}u(z)\, dm(z) + \int_{(x,y]} u(z)\, dk(z), \quad x, y \in I.$$

So we sometimes use the symbol $\mathcal{G}u = (dD_s u - u\, dk)/dm$. Following McKean [4] (see also Section 4.11 of [2]), we can define the fundamental solution $p(t, x, y)$ of the following equation.

$$\frac{\partial}{\partial t} p(t, x, y) = \mathcal{G}p(t, x, y), \quad t > 0,\ x, y \in I,$$

where $\mathcal{G}$ is applied to $x$ or $y$.

It is known that $p(t, x, y)$ satisfies the following properties:

$$\begin{aligned}
&0 < p(t, x, y) = p(t, y, x) \text{ is continuous on } I \times I \times (0, \infty),\\
&p(s+t, x, y) = \int_I p(s, x, z) p(t, z, y)\, dm(z), \quad s, t > 0,\ x, y \in I,\\
&p(t, l_i, y) = 0, \quad t > 0,\ y \in I, \quad \text{if } l_i \text{ is not entrance},\\
&D_s p(t, l_i, y) = 0, \quad t > 0,\ y \in I, \quad \text{if } l_i \text{ is entrance},
\end{aligned}$$

where $D_s p(t, x, y) = \lim_{\varepsilon \downarrow 0} \{p(t, x+\varepsilon, y) - p(t, x, y)\} / \{s(x+\varepsilon) - s(x)\}$. It is also known that there exists a one-dimensional generalized diffusion process (ODGDP for brief) $\mathbb{D} = [X(t) : t \geq 0,\ P_x : x \in I^*]$ such that

$$P_x(X(t) \in E) = \int_E p(t, x, y)\, dm(y), \quad t > 0,\ x \in I^*,\ E \in \mathcal{B}(I^*).$$

By this reason, $p(t, x, y)$ is sometimes called the transition probability density with respect to $m$. The state of boundaries, that is, $(s, m, k)$ -regular, exit, entrance, and natural, suggest the behavior of the sample paths of $\mathbb{D}$ having the ODGDO $\mathcal{G}$ with $(s, m, k)$ as the generator (see [2]). For $\beta \geq 0$ let $\mathcal{H}_{s,m,k,\beta}$ be the set of all positive functions $h_\beta$ satisfying

$$\begin{aligned}
h_\beta(x) =& h_\beta(c_o) + D_s h_\beta(c_o)\{s(x) - s(c_o)\}\\
&+ \int_{(c_o,x]} \{s(x) - s(y)\} h_\beta(y) \{\beta dm(y) + dk(y)\}, \quad x \in I.
\end{aligned}$$

We call $h_\beta$ a $\beta$ harmonic function for $\mathcal{G}$. For $h \in \mathcal{H}_{s,m,k,\beta}$, we set

$$s_h(x) = \int_{(c_o,x]} h(y)^{-2}\, ds(y), \tag{1.2}$$

$$m_h(x) = \int_{(c_o,x]} h(y)^2\, dm(y). \tag{1.3}$$

$$p_h(t, x, y) = e^{-\beta t} p(t, x, y)/h(x)h(y).$$

Let $\mathcal{G}_h$ be an ODGDO with $(s_h, m_h, 0)$, where 0 denotes the null measure. Let $\mathbb{D}_h$ be an ODGDP with $\mathcal{G}_h$ as the generator. Then $p_h(t, x, y)$ is the transition probability density of $\mathbb{D}_h$ with respect to $m_h$. We call $\mathbb{D}_h$ a harmonic transform of $\mathbb{D}$. In this paper we study state of boundaries for $\mathbb{D}_h$. Our main result is as follows.

**Theorem 1.1** *Let $h \in \mathcal{H}_{s,m,k,\beta}$ and $i = 1, 2$.*
*(i) Suppose that $l_i$ is $(s, m, k)$ -regular or exit. If $h(l_i) = 0$, then $l_i$ is $(s_h, m_h, 0)$ -entrance. If $0 < h(l_i) < \infty$, then $l_i$ is $(s_h, m_h, 0)$ -regular or exit according to $l_i$ being $(s, m, k)$ -regular or exit.*
*(ii) Suppose that $l_i$ is $(s, m, k)$ -entrance. If $0 < h(l_i) < \infty$, then $l_i$ is $(s_h, m_h, 0)$ -entrance. If $h(l_i) = \infty$, then $l_i$ is $(s_h, m_h, 0)$ -regular or exit according to $|m_h(l_i)| < \infty$ or $|m_h(l_i)| = \infty$.*
*(iii) Suppose that $l_i$ is $(s, m, k)$ -natural. If $h(l_i) = 0$, then $l_i$ is $(s_h, m_h, 0)$ -entrance or natural according to $J_{m_h,s_h}(l_i) < \infty$ or $J_{m_h,s_h}(l_i) = \infty$. If $h(l_i) = \infty$, then $l_i$ is $(s_h, m_h, 0)$ -regular, exit, or natural according to $|m(l_i)| < \infty$, $|m(l_i)| = \infty$ and $J_{s_h,m_h}(l_i) < \infty$, or $|m(l_i)| = \infty$ and $J_{s_h,m_h}(l_i) = \infty$.*

The statements of the theorem are tabled as follows.

Table 1

<table>
<tr><th></th><th>$h(l_i) = 0$</th><th>$h(l_i) \in (0, \infty)$</th><th>$h(l_i) = \infty$</th></tr>
<tr><td>$(s, m, k)$ -regular</td><td>$(s_h, m_h, 0)$ -entrance Ex. 3.1</td><td>$(s_h, m_h, 0)$ -regular Ex. 3.2</td><td>$\emptyset$</td></tr>
<tr><td>$(s, m, k)$ -exit</td><td>$(s_h, m_h, 0)$ -entrance Ex. 3.3</td><td>$(s_h, m_h, 0)$ -exit Ex. 3.4</td><td>$\emptyset$</td></tr>
<tr><td rowspan="2">$(s, m, k)$ -entrance</td><td rowspan="2">$\emptyset$</td><td rowspan="2">$(s_h, m_h, 0)$ -entrance Ex. 3.5</td><td>$(s_h, m_h, 0)$ -regular if $|m_h(l_i)| < \infty$ Ex. 3.2</td></tr>
<tr><td>$(s_h, m_h, 0)$ -exit if $|m_h(l_i)| = \infty$ Ex. 3.5</td></tr>
<tr><td rowspan="3">$(s, m, k)$ -natural</td><td rowspan="2">$(s_h, m_h, 0)$ -entrance if $J_{m_h,s_h}(l_i) < \infty$ Ex. 3.6</td><td rowspan="3">$\emptyset$</td><td>$(s_h, m_h, 0)$ -regular if $|m_h(l_i)| < \infty$ Ex. 3.7</td></tr>
<tr><td>$(s_h, m_h, 0)$ -exit if $|m(l_1)| = \infty$, $J_{s_h,m_h}(l_i) < \infty$ Ex. 3.6</td></tr>
<tr><td>$(s_h, m_h, 0)$ -natural if $J_{m_h,s_h}(l_i) = \infty$ Ex. 3.4, Ex. 3.5 Ex. 3.6, Ex. 3.7</td><td>$(s_h, m_h, 0)$ -natural if $|m(l_1)| = \infty$, $J_{s_h,m_h}(l_i) = \infty$ Ex. 3.1, Ex. 3.3 Ex. 3.5, Ex. 3.6</td></tr>
</table>

The symbol $\emptyset$ of the table means that there don't exist any $\beta$ harmonic functions for $\mathcal{G}$ (see Lemma 2.1 below). We exhibit examples for each cases of the table in Section 3. Example 3.1 etc. are abbreviated as Ex.3.1 etc., respectively.

In [3] Maeno treated harmonic transforms different from ours. More precisely, let $s$ and $m$ be the scale function and speed measure on $I$. Let $\mathcal{M}_s^o$ be the set of all positive continuous functions $h$ on $I$ such that $h$ has the right derivative $D_s h$ which is right continuous and nonincreasing. For $h \in \mathcal{M}_s^o$ we consider $s_h$ and $m_h$ given by (1.2) and (1.3), respectively. Further set $k_h(x) = -\int_{(c_o,x]} hdD_sh(x)$. Let $\mathcal{G}_h^o$ be an ODGDO with $(s_h, m_h, k_h)$. She discusses the state of boundaries for $\mathbb{D}_h^o$ having the ODGDO $\mathcal{G}_h^o$ as the generator. Since $\mathcal{H}_{s,m,k,\beta} \cap \mathcal{M}_s^o = \emptyset$ if $k \neq 0$ or $\beta > 0$, we cannot derive Theorem 1.1 from her results. However there is a relation between our harmonic transform and Maeno's harmonic transform. We discuss this relation in [5].

# 2 Proof of main theorem

In this section we prove Theorem 1.1 for $l_1$. First we summarize some properties of $\beta$ harmonic functions.

**Lemma 2.1** ( [2], [6]) *Let $h \in \mathcal{H}_{s,m,k,\beta}$.*
(i) *For $l_1 < x < y < l_2$,*

$$D_sh(x) \leq \frac{h(y) - h(x)}{s(y) - s(x)} \leq D_sh(y).$$

(ii) *Suppose $l_1$ is regular or exit. Then $0 \leq h(l_1) < \infty$. If $h(l_1) = 0$, $h(x) \leq D_sh(x)(s(x) - s(l_1))$.*
(iii) *Suppose $l_1$ is entrance. Then $0 < h(l_1) \leq \infty$. If $h(l_1) = \infty$, then $D_sh(l_1) \in [-\infty, 0)$, $|s_h(l_1)| < \infty$, and $\int_{(l_1,c_o]} h(y)\, dm(y) < \infty$.*
(iv) *Suppose $l_1$ is natural. Then $h(l_1) = 0$, or $h(l_1) = \infty$. If $h(l_1) = \infty$, $|s_h(l_1)| < \infty$.*

**Remark 2.2** Suppose that $h \in \mathcal{H}_{s,m,k,\beta}$ and $0 < h(l_1) < \infty$. Then

$$0 < \lim_{x \downarrow l_1} \frac{s_h(x)}{s(x)} < \infty, \quad 0 < \lim_{x \downarrow l_1} \frac{m_h(x)}{m(x)} < \infty.$$

The statements of Theorem 1.1 (i) and (ii) corresponding to $0 < h(l_1) < \infty$ are derived from Remark 2.2. We divide the proof of the theorem into three cases for expect $0 < h(l_1) < \infty$. In the following we fix $s$, $m$, $k$, $\beta$, and $h \in \mathcal{H}_{s,m,k,\beta}$.

## 2.1 The case that $l_1$ is $(s, m, k)$ -regular or exit

Suppose that $l_1$ is $(s, m, k)$ -regular or exit. Then $0 \leq h(l_1) < \infty$. If $h(l_1) = 0$, by means of Lemma 2.1, there is an $x_0 \in I$ such that

$$h(x) \leq (s(x) - s(l_1))D_sh(x_0), \quad l_1 < x < x_0.$$

Therefore

$$\int_{(l_1,x_0]} h^{-2}(x)\, ds(x) \geq (D_sh(x_0))^{-2} \int_{(l_1,x_0]} \frac{ds(x)}{(s(x) - s(l_1))^2} = \infty.$$

Hence $s_h(l_1) = -\infty$. Furthermore

$$\int_{(l_1,x_0]} h^2(x)\, dm(x) \int_{(x,x_0]} h^{-2}(y)\, ds(y) \leq \int_{(l_1,x_0]} dm(x) \int_{(x,x_0]} ds(y) < \infty,$$

which shows that $l_1$ is $(s_h, m_h, 0)$ -entrance.

## 2.2 The case that $l_1$ is $(s,m,k)$ -entrance

Suppose that $l_1$ is $(s,m,k)$ -entrance. If $h(x)=\infty$, then we have

$$\lim_{x\to l_1}\frac{\int_{(l_1,x]}h^{-2}(y)\,ds(y)}{h^{-1}(x)}=-\lim_{x\to l_1}\frac{1}{D_sh(x)}=-\frac{1}{D_sh(l_1)}.$$

We note that $D_sh(l_1)\in[-\infty,0)$ by means of Lemma 2.1. Hence there are $x_0\in I$ and a positive constant $C$ such that $\int_{(l_1,x]}h^{-2}(y)\,ds(y)\le Ch^{-1}(x)$, $l_1<x<x_0$. Combing this with $|s_h(l_1)|<\infty$, we find

$$\begin{aligned}\int_{(l_1,x_0]}h^{-2}(y)\,ds(y)\int_{(y,x_0]}h^2(x)\,dm(x)&=\int_{(l_1,x_0]}h^2(y)\,dm(y)\int_{(l_1,y]}h^{-2}(x)\,ds(x)\\&\le C\int_{(l_1,c]}h(y)\,dm(y)<\infty.\end{aligned}$$

Thus we have that $l_1$ is $(s_h,m_h,0)$ -regular (resp. -exit) if $|m_h(l_1)|<\infty$ (resp. $|m_h(l_1)|=\infty$).

## 2.3 The case that $l_1$ is $(s,m,k)$ -natural

Suppose that $l_1$ is $(s,m,k)$ -natural. Then $h(l_1)=0$ or $h(l_1)=\infty$.

Suppose that $h(l_1)=0$. If $s(l_1)=-\infty$, for any $M>0$ there exists an $x_0$ such that $\frac{1}{h(x)}>M$, $l_1<x\le x_0$. We have

$$\int_{(l_1,x_0]}h^{-2}(x)\,ds(x)\ge M^2\int_{(l_1,x_0]}ds(x)=\infty.$$

Hence we have $s_h(l_1)=-\infty$. If $s(l_1)>-\infty$, there exist an $x_1$ and a positive constant $C$ such that $h(x)\le C(s(x)-s(l_1))$ for $l_1<x\le x_1$. Therefore

$$\int_{(l_1,x_1]}h^{-2}(x)\,ds(x)\ge\frac{1}{C}\int_{(l_1,x_1]}\frac{ds(x)}{(s(x)-s(l_1))^2}=\infty,$$

Then we have $s_h(l_1)=-\infty$. Thus $l_1$ is $(s_h,m_h,0)$ -entrance or natural according to $J_{m_h,s_h}(l_1)<\infty$ or $J_{m_h,s_h}(l_1)=\infty$.

If $h(l_1)=\infty$, then

$$\int_{(l_1,c]}h^{-2}(y)\,ds(y)<\infty,$$

by means of Lemma 2.1. Therefore $l_1$ is $(s_h,m_h,0)$ -regular, exit, or natural according to $|m_h(l_i)|<\infty$, $|m_h(l_i)|=\infty$ and $J_{s_h,m_h}(l_1)<\infty$, or $|m_h(l_i)|<\infty$ and $J_{s_h,m_h}(l_1)=\infty$.

# 3 Examples

In this section we give each examples in Table 1.

**Example 3.1** Consider the generator

$$\mathcal{G} = x^2(x^2-1)\frac{d^2}{dx^2} + 2x^3\frac{d}{dx} - x^2(x^2-1)^{-1},$$

on $(1,\infty)$. The scale function, speed measure, and killing measure are given by

$$ds(x) = (x^2-1)^{-1}\,dx,\ dm(x) = x^{-2}\,dx,\ dk(x) = (x^2-1)^{-1}\,dx.$$

The end points 1 and $\infty$ are $(s,m,k)$ -natural and $(s,m,k)$ -regular, respectively.

Set $h(x) = (x^2-1)^{-\frac{1}{2}}$. $h(x)$ is a 0 harmonic function for $\mathcal{G}$. We obtain the transformed scale function , the transformed speed measure, and the transformed generator as follows.

$$ds_h(x) = dx,\quad dm_h(x) = x^{-2}(x^2-1)^{-1}\,dx,\quad \mathcal{G}_h = x^2(x^2-1)\frac{d^2}{dx^2}.$$

Since $J_{s_h,m_h}(1) = \infty$ and $J_{m_h,s_h}(1) = \infty$, the end point 1 is $(s_h,m_h,0)$ -natural. Since $|s_h(\infty)| = \infty$ and $J_{m_h,s_h}(\infty) < \infty$, the end point $\infty$ is $(s_h,m_h,0)$ -entrance.

**Example 3.2** Consider the generator

$$\mathcal{G} = e^{\gamma x}\frac{d^2}{dx^2} - \kappa,$$

on $(0,\infty)$, where $\gamma > 0$ and $\kappa > 0$. The scale function, speed measure, and killing measure are given by

$$ds(x) = dx,\ dm(x) = e^{-\gamma x}\,dx,\ dk(x) = \kappa e^{-\gamma x}\,dx.$$

The end points 0 and $\infty$ are $(s,m,k)$ -regular and $(s,m,k)$ -entrance, respectively.

For $\lambda \geq 0$, set $h(x) = K_0(\frac{2\sqrt{\lambda+\kappa}}{\gamma}e^{-\frac{\gamma x}{2}})$. $h(x)$ is a $\lambda$ harmonic function for $\mathcal{G}$. We obtain the transformed scale function, the transformed speed measure, and the transformed generator as follows.

$$\begin{aligned} ds_h(x) =& K_0^{-2}\left(\frac{2\sqrt{\lambda+\kappa}}{\gamma}e^{-\frac{\gamma x}{2}}\right)dx,\quad dm_h(x) = K_0^2\left(\frac{2\sqrt{\lambda+\kappa}}{\gamma}e^{-\frac{\gamma x}{2}}\right)e^{-\gamma x}\,dx,\\ \mathcal{G}_h =& e^{\gamma x}\frac{d^2}{dx^2} + 2\sqrt{\lambda+\kappa}\,e^{\frac{\gamma x}{2}}\frac{K_1(\frac{2\sqrt{\lambda+\kappa}}{\gamma}e^{-\frac{\gamma x}{2}})}{K_0(\frac{2\sqrt{\lambda+\kappa}}{\gamma}e^{-\frac{\gamma x}{2}})}\frac{d}{dx}. \end{aligned}$$

Since $|s_h(0)| < \infty$ and $|m_h(0)| < \infty$, the end point 0 is $(s_h,m_h,0)$ -regular. Since $|s_h(\infty)| < \infty$ and $|m_h(\infty)| < \infty$, the end point $\infty$ is $(s_h,m_h,0)$ -regular.

**Example 3.3** Consider the generator

$$\mathcal{G} = x(x^2-1)\frac{d^2}{dx^2} + 2x^2\frac{d}{dx} - x(x^2-1)^{-1},$$

on $(1,\infty)$. The scale function, speed measure, and killing measure are given by

$$ds(x) = (x^2-1)^{-1}\,dx,\ dm(x) = x^{-1}\,dx,\ dk(x) = (x^2-1)^{-1}\,dx.$$

The end points 1 and $\infty$ are $(s,m,k)$ -natural and $(s,m,k)$ -exit, respectively.

Set $h(x) = (x^2-1)^{-\frac{1}{2}}$. $h(x)$ is a 0 harmonic function for $\mathcal{G}$. We obtain the transformed scale function, the transformed speed measure, and the transformed generator as follows.

$$ds_h(x) = dx, \quad dm_h(x) = x^{-1}(x^2-1)^{-1}\,dx, \quad \mathcal{G}_h = x(x^2-1)\frac{d^2}{dx^2}.$$

Since $|m_h(1)| = \infty$ and $J_{s_h,m_h}(1) = \infty$, the end point 1 is $(s_h, m_h, 0)$ -natural. Since $|s_h(\infty)| = \infty$ and $J_{m_h,s_h}(\infty) < \infty$, the end point $\infty$ is $(s_h, m_h, 0)$ -entrance.

**Example 3.4** Consider the generator

$$\mathcal{G} = x^{\frac{3}{2}}\frac{d^2}{dx^2} + \frac{2x^{\frac{3}{2}}}{(x+1)(\log(x+1)+1)}\frac{d}{dx} - \frac{x^{\frac{3}{2}}}{(x+1)^2(\log(x+1)+1)},$$

on $(0,\infty)$. The scale function, speed measure, and killing measure are given by

$$\begin{aligned} ds(x) =&(\log(x+1)+1)^{-2}\,dx, \\ dm(x) =&(\log(x+1)+1)^2 x^{-\frac{3}{2}}\,dx, \\ dk(x) =&(\log(x+1)+1)(x+1)^{-2}\,dx. \end{aligned}$$

The end points 0 and $\infty$ are $(s,m,k)$ -exit and $(s,m,k)$ -natural, respectively.

Set $h(x) = (\log(x+1)+1)^{-1}$. $h(x)$ is a 0 harmonic function for $\mathcal{G}$. We obtain the transformed scale function, the transformed speed measure, and the transformed generator as follows.

$$ds_h(x) =dx, \quad dm_h(x) = x^{-\frac{3}{2}}\,dx, \quad \mathcal{G}_h = x^{\frac{3}{2}}\frac{d^2}{dx^2}.$$

Since $|m_h(0)| = \infty$ and $J_{s_h,m_h}(0) < \infty$, the end point 0 is $(s_h, m_h, 0)$ -exit. Since $|s_h(\infty)| = \infty$ and $J_{m_h,s_h}(\infty) = \infty$, the end point $\infty$ is $(s_h, m_h, 0)$ -natural.

**Example 3.5** Consider the generator

$$\mathcal{G} = \frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \gamma,$$

on $(0,\infty)$, where $\gamma > 0$. The scale function, speed measure, and killing measure are given by

$$ds(x) = x^{-1}\,dx, \; dm(x) = 2x\,dx, \; dk(x) = 2\gamma x\,dx.$$

The end points 0 and $\infty$ are $(s,m,k)$ -entrance and $(s,m,k)$ -natural, respectively.

We consider two harmonic transforms for $\mathcal{G}$.

For $\lambda \geq 0$, set $\phi(x) = x^{-1}\sinh(x\sqrt{2(\lambda+\gamma)})$. $\phi(x)$ is a $\lambda$ harmonic function for $\mathcal{G}$. We obtain the transformed scale function, the transformed speed measure, and the transformed generator as follows.

$$\begin{aligned} ds_\phi(x) =&x\sinh^{-2}(x\sqrt{2(\lambda+\gamma)})\,dx, \quad dm_\phi(x) = 2x^{-1}\sinh^2(x\sqrt{2(\lambda+\gamma)})\,dx, \\ \mathcal{G}_\phi =&\frac{1}{2}\frac{d^2}{dx^2} + \left(\frac{1}{x} - \frac{1}{x}\sinh(x\sqrt{2(\lambda+\gamma)}) + \tanh^{-1}(x\sqrt{2(\lambda+\gamma)})\right)\frac{d}{dx}. \end{aligned}$$

Since $|s_\phi(0)| = \infty$ and $J_{m_\phi,s_\phi}(0) < \infty$, the end point 0 is $(s_\phi, m_\phi, 0)$ -entrance. Since $|m_\phi(\infty)| = \infty$ and $J_{s_\phi,m_\phi}(\infty) = \infty$, the end point $\infty$ is $(s_\phi, m_\phi, 0)$ -natural.

For $\lambda \geq 0$, set $\psi(x) = x^{-1}e^{-x\sqrt{2(\lambda+\gamma)}}$. $\psi(x)$ is a $\lambda$ harmonic function for $\mathcal{G}$. We obtain the transformed scale function, the transformed speed measure, and the transformed generator as follows.

$$ds_\psi(x) = xe^{2x\sqrt{2(\lambda+\gamma)}}\,dx, \quad dm_\psi(x) = 2xe^{-2x\sqrt{2(\lambda+\gamma)}}\,dx, \quad \mathcal{G}_\psi = \frac{1}{2}\frac{d^2}{dx^2} + \frac{d}{dx}.$$

Since $|m_\psi(0)| = \infty$ and $J_{s_\psi,m_\psi}(0) < \infty$, the end point 0 is $(s_\psi, m_\psi, 0)$ -exit. Since $|s_\psi(\infty)| = \infty$ and $J_{m_\psi,s_\psi}(\infty) = \infty$, the end point $\infty$ is $(s_\psi, m_\psi, 0)$ -natural.

**Example 3.6** Consider the generator

$$\mathcal{G} = \frac{1}{2}\frac{d^2}{dx^2} - \frac{\gamma^2 - 2^{-2}}{2x^2},$$

on $(0,\infty)$, where $|\gamma| > \frac{1}{2}$. The scale function, speed measure, and killing measure are given by

$$ds(x) = dx, \; dm(x) = 2\,dx, \; dk(x) = (\gamma^2 - 2^{-2})x^{-2}\,dx.$$

The end points 0 and $\infty$ are $(s, m, k)$ -natural.

We consider two harmonic transforms for $\mathcal{G}$.

For $\lambda > 0$, set $\phi(x) = \sqrt{x}I_\gamma(x\sqrt{2\lambda})$. $\phi(x)$ is a $\lambda$ harmonic function for $\mathcal{G}$. We obtain the transformed scale function, the transformed speed measure, and the transformed generator as follows.

$$\begin{aligned} ds_\phi(x) =& x^{-1}I_\gamma^{-2}(x\sqrt{2\lambda})\,dx, \quad dm_\phi(x) = 2xI_\gamma^2(x\sqrt{2\lambda})\,dx, \\ \mathcal{G}_\phi =& \frac{1}{2}\frac{d^2}{dx^2} + \left(\frac{1}{2x} + \frac{\gamma}{x\sqrt{2\lambda}} + \frac{I_{\gamma+1}(x\sqrt{2\lambda})}{I_\gamma(x\sqrt{2\lambda})}\right)\frac{d}{dx}. \end{aligned}$$

Since $|s_\phi(0)| = \infty$ and $J_{m_\phi,s_\phi}(0) < \infty$, the end point 0 is $(s_\phi, m_\phi, 0)$ -entrance. Since $|m_\phi(\infty)| = \infty$ and $J_{s_\phi,m_\phi}(\infty) = \infty$, the end point $\infty$ is $(s_h, m_h, 0)$ -natural.

For $\lambda > 0$ set $\psi(x) = \sqrt{x}K_\gamma(x\sqrt{2\lambda})$. $\psi(x)$ is a $\lambda$ harmonic function for $\mathcal{G}$. We obtain the transformed scale function and the transformed speed measure as follows.

$$\begin{aligned} ds_\psi(x) =& x^{-1}K_\gamma^{-2}(x\sqrt{2\lambda})\,dx, \quad dm_\psi(x) = 2xK_\gamma^2(x\sqrt{2\lambda})\,dx, \\ \mathcal{G}_\psi =& \frac{1}{2}\frac{d^2}{dx^2} + \left(\frac{1}{2x} + \frac{\gamma}{x\sqrt{2\lambda}} - \frac{K_{\gamma+1}(x\sqrt{2\lambda})}{K_\gamma(x\sqrt{2\lambda})}\right)\frac{d}{dx}. \end{aligned}$$

Since $|m_\psi(0)| = \infty$ and $J_{s_\psi,m_\psi}(0) < \infty$, the end point 0 is $(s_\psi, m_\psi, 0)$ -exit. Since $|s_\psi(\infty)| = \infty$ and $J_{m_\psi,s_\psi}(\infty) = \infty$, the end point $\infty$ is $(s_h, m_h, 0)$ -natural.

**Example 3.7** Consider the generator

$$\mathcal{G} = \frac{x^4}{2}\frac{d^2}{dx^2} + \frac{x^3}{2}\frac{d}{dx} - \frac{1}{8}x^2,$$

on $(0,\infty)$. The scale function, speed measure, and killing measure are given by

$$ds(x) = x^{-1}dx, \; dm(x) = 2x^{-3}\,dx, \; dk(x) = \frac{1}{4}x^{-1}\,dx.$$

The end points 0 and $\infty$ are $(s, m, k)$ -natural.

Set $h(x) = x^{\frac{1}{2}}$. $h(x)$ is a 0 harmonic function for $\mathcal{G}$. We obtain the transformed scale function, the transformed speed measure, and the transformed generator as follows.

$$ds_h(x) = x^{-2}\,dx, \quad dm_h(x) = 2x^{-2}\,dx, \quad \mathcal{G}_h = \frac{x^4}{2}\frac{d^2}{dx^2} + x^3\frac{d}{dx}.$$

Since $|s_h(0)| = \infty$ and $|m_h(0)| = \infty$, the end point 0 is $(s_h, m_h, 0)$ -natural. Since $|s_h(\infty)| < \infty$ and $|m_h(\infty)| < \infty$, the end point $\infty$ is $(s_h, m_h, 0)$ -regular.

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# State of boundaries for harmonic transforms of one-dimensional generalized diffusion processes

TAKEMURA Tomoko

We consider a one-dimensional generalized diffusion operator $\mathcal{G}$ represented by triplet of Borel measures and a harmonic transform $\mathcal{G}_h$ of $\mathcal{G}$, where $h$ is a harmonic function for $\mathcal{G}$. Specially we treat an operator with killing measure which is not null measure. We consider the state of boundaries for the one-dimensional generalized diffusion process $\mathbb{D}_h$ with $\mathcal{G}_h$ as the generator. State of boundaries for $\mathbb{D}_h$ may be different from those for $\mathbb{D}$ which is a one-dimensional generalized process with $\mathcal{G}$ as the generator. We characterize the state of boundaries for $\mathbb{D}_h$ in terms of the Borel measures and a harmonic function for $\mathcal{G}$. After we prove our main theorem, we give some examples.