

Conditional diffusion models

MAENO Miyuki*

1 Introduction

In this paper we will be concerned with one-dimensional diffusion processes conditioned by hitting times. More precisely, let $\mathbb{D} = [X(t) : t \geq 0, P_x : x \in I]$ be a one-dimensional diffusion process on a bounded interval I with the end points a, b ($a < b$) and consider the following limit distribution if it exists:

$$P^*(t, x, E) \equiv \lim_{n \rightarrow \infty} P_x(X(t) \in E | \sigma_{b_n} < \sigma_{a_n}), \quad (1.1)$$

for $t > 0, x \in (a, b)$ and $E \in \mathcal{B}((a, b))$, where $\{a_n\}, \{b_n\}$ are sequences satisfying $a_n \downarrow a, b_n \uparrow b$ as $n \rightarrow \infty$, and σ_c denotes the first hitting time to a point $c \in I$, that is, $\sigma_c = \inf\{t > 0; X(t) = c\}$. If both end points are accessible, then the limit distribution $P^*(t, x, E)$ exists and coincides with

$$P_x(X(t) \in E | \sigma_b < \sigma_a). \quad (1.2)$$

Otherwise, it may not exist. So we suppose that $\lim_{x \uparrow b} s(x)$ is finite for the scale function s of \mathbb{D} . Under this assumption, we obtain the limit distribution $P^*(t, x, E)$. Obviously $P^*(t, x, E)$ is a transition probability and hence it leads us to a Markov process $\mathbb{D}^* = [X^*(t) : t \geq 0, P_x^* : x \in (a, b)]$. We will show that \mathbb{D}^* is a diffusion process and give a characterization of \mathbb{D}^* .

The distributions like (1.2) sometimes appear in the theory of population genetics. Namely, let $\mathbb{D} = [X(t) : t \geq 0, P_x : x \in I]$ be a diffusion approximation of gene frequency process. Accordingly the end points of I are 0 and 1. If the end points 0 and 1 are accessible, then the conditional distribution $P_x(X(t) \in E | \sigma_1 < \sigma_0)$ stands for the distribution of frequency of a certain allele under the condition that it reaches fixation before it disappears from the population. Such distributions and related topics are already studied by Ewens [1], [2] and Karlin and Taylor [8] for the case that the end points are accessible. However the end points 0 and 1 may not be accessible. Our argument in the preceding paragraph is also available in such cases. Actually we apply our general results to population genetics to obtain some results including those discussed by Ewens [1], [2] and Karlin and Taylor [8].

* School of Interdisciplinary Research of Scientific Phenomena and Information

2 Conditional diffusion processes

Let us consider the one-dimensional diffusion process $\mathbb{D} = [X(t) : t \geq 0, P_x : x \in I]$. Note that $P_x(X(0) = x) = 1$ for $x \in I$. We may assume that the end points of I are 0 and 1 without loss of generality. We suppose that the generator is given by

$$L = \frac{1}{2}a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}, \quad (2.1)$$

where $a(x), b(x) \in C((0, 1))$ and $a(x)$ is positive on $(0, 1)$. Let us fix a point $c \in (0, 1)$ arbitrarily and set

$$\begin{aligned} s_o(x) &= \exp \left\{ -2 \int_c^x \frac{b(y)}{a(y)} dy \right\}, & m_o(x) &= \frac{2}{a(x)} \exp \left\{ 2 \int_c^x \frac{b(y)}{a(y)} dy \right\}, \\ s(x) &= \int_c^x s_o(y) dy, & m(x) &= \int_c^x m_o(y) dy. \end{aligned}$$

Then the generator L is reduced to $L = \frac{d}{dm} \frac{d}{ds}$. Following [7], we call s and dm the scale function and the speed measure, respectively. We introduce the following two quantities:

$$\begin{aligned} I(x) &= \int_c^x s_o(y) dy \int_c^y m_o(z) dz, \\ J(x) &= \int_c^x m_o(y) dy \int_c^y s_o(z) dz, \end{aligned}$$

for $0 < x < 1$. Following [3], we say that the state of the end point ℓ ($= 0$ or 1) is

regular	if	$\lim_{x \rightarrow \ell} I(x) < \infty, \lim_{x \rightarrow \ell} J(x) < \infty,$
exit	if	$\lim_{x \rightarrow \ell} I(x) < \infty, \lim_{x \rightarrow \ell} J(x) = \infty,$
entrance	if	$\lim_{x \rightarrow \ell} I(x) = \infty, \lim_{x \rightarrow \ell} J(x) < \infty,$
natural	if	$\lim_{x \rightarrow \ell} I(x) = \infty, \lim_{x \rightarrow \ell} J(x) = \infty.$

We assume that the end point ℓ is included in I if it is not natural. It is known that

$$P_x(\sigma_\eta < \sigma_\xi) = \frac{s(x) - s(\xi)}{s(\eta) - s(\xi)}, \quad \xi \wedge \eta < x < \xi \vee \eta, \quad x, \xi, \eta \in I, \quad (2.2)$$

where $\xi \wedge \eta = \min\{\xi, \eta\}$ and $\xi \vee \eta = \max\{\xi, \eta\}$ (cf. [7, p.112, (16)]). Now we assume that

$$s(1-) < \infty. \quad (2.3)$$

The assumption (2.3) implies that the end point 1 is regular, exit, or natural. Under the assumption (2.3), there exists the limit

$$\begin{aligned} \pi(x) &\equiv \lim_{n \rightarrow \infty} P_x(\sigma_{\eta_n} < \sigma_{\xi_n}) \\ &= \begin{cases} \frac{s(x) - s(0+)}{s(1-) - s(0+)} & \text{if } s(0+) > -\infty, \\ 1 & \text{if } s(0+) = -\infty, \end{cases} \end{aligned} \quad (2.4)$$

for any sequences $\{\xi_n\}$, $\{\eta_n\}$ satisfying $\xi_n \downarrow 0$, $\eta_n \uparrow 1$ as $n \rightarrow \infty$.

It is also known that the transition probability has the density $p(t, x, y)$ with respect to the speed measure, that is,

$$P_x(X(t) \in E) = \int_E p(t, x, y) m_o(y) dy, \quad t > 0, x \in I, E \in \mathcal{B}(I).$$

The density $p(t, x, y)$ is positive and continuous in $(t, x, y) \in (0, \infty) \times I \times I$ and symmetric in x, y . Moreover $p(t, x, y)$ satisfies the equation

$$\frac{\partial}{\partial t} p(t, x, y) = L_x p(t, x, y) = L_y p(t, x, y), \quad t > 0, 0 < x < 1, 0 < y < 1, \quad (2.5)$$

where L_x [resp. L_y] means the operator L applied to x [resp. y] (cf. [7, Sect. 4.11]).

Next we introduce the stopped diffusion process $\mathbb{D}_o \equiv [X_o(t) : t \geq 0, P_x : x \in (0, 1)]$, where

$$X_o(t) = \begin{cases} X(t \wedge \sigma_0) & \text{if the end point 0 is regular and reflecting,} \\ X(t) & \text{otherwise.} \end{cases}$$

Let $p_o(t, x, y)$ be the transition probability density of \mathbb{D}_o . We note that $p_o(t, x, y)$ coincides with $p(t, x, y)$ except the case that the end point 0 is regular and reflecting.

Now we state our main results.

Theorem 2.1 *Suppose (2.3) and the following:*

$$\text{if the end point 1 is regular, then it is absorbing.} \quad (2.6)$$

Then there exists the limit of the following conditional distribution:

$$\begin{aligned} P^*(t, x, E) &\equiv \lim_{n \rightarrow \infty} P_x(X(t) \in E | \sigma_{\eta_n} < \sigma_{\xi_n}), \\ &t \geq 0, x \in (0, 1), E \in \mathcal{B}((0, 1)), \end{aligned} \quad (2.7)$$

for any sequences $\{\xi_n\}$, $\{\eta_n\}$ satisfying $\xi_n \downarrow 0$, $\eta_n \uparrow 1$ as $n \rightarrow \infty$. The limit $P^*(t, x, y)$ is independent of choice of sequences $\{\xi_n\}$ and $\{\eta_n\}$. The set function $P^*(t, x, \cdot)$ is absolutely continuous and it holds that

$$P^*(t, x, E) = \int_E p_o(t, x, y) \frac{\pi(y)}{\pi(x)} m_o(y) dy, \quad t > 0, x \in (0, 1), E \in \mathcal{B}((0, 1)). \quad (2.8)$$

Next we show that $P^*(t, x, E)$ is the transition probability of a diffusion process on $(0, 1)$. We set

$$p^*(t, x, y) = p_o(t, x, y) / \pi(x) \pi(y), \quad t > 0, x, y \in (0, 1), \quad (2.9)$$

$$s_o^*(x) = s_o(x) / \pi(x)^2, \quad m_o^*(x) = \pi(x)^2 m_o(x), \quad x \in (0, 1). \quad (2.10)$$

By means of (2.8),

$$P^*(t, x, E) = \int_E p^*(t, x, y) m_o^*(y) dy, \quad t > 0, x \in (0, 1), E \in \mathcal{B}((0, 1)). \quad (2.11)$$

We also define the operator L^* by

$$L^* = \frac{1}{2} a^*(x) \frac{d^2}{dx^2} + b^*(x) \frac{d}{dx} = \frac{d}{dm^*} \frac{d}{ds^*}, \quad (2.12)$$

where

$$a^*(x) = a(x); \quad b^*(x) = \begin{cases} b(x) + \frac{s_o(x)}{s(x) - s(0+)} a(x) & \text{if } s(0+) > -\infty, \\ b(x) & \text{if } s(0+) = -\infty, \end{cases}$$

$$s^*(x) = \int_c^x s_o^*(y) dy, \quad m^*(x) = \int_c^x m_o^*(y) dy.$$

Theorem 2.2 Assume (2.3) and (2.6).

(i) $p^*(t, x, y)$ satisfies the following equation

$$\frac{\partial}{\partial t} p^*(t, x, y) = L_x^* p^*(t, x, y) = L_y^* p^*(t, x, y), \quad t > 0, 0 < x, y < 1, \quad (2.13)$$

where L_x^* [resp. L_y^*] means the operator L^* applied to x [resp. y].

(ii) The function $P^*(t, x, E)$ is the transition probability of a diffusion process $\mathbb{D}^* = [X^*(t) : t \geq 0, P_x^* : x \in (0, 1)]$ whose generator is given by L^* .

(iii) If the end point 0 is regular, exit or entrance for the process \mathbb{D} , then it is entrance for the process \mathbb{D}^* . If the end point 0 is natural for the process \mathbb{D} , then it is also natural for the process \mathbb{D}^* . The state of end point 1 for the process \mathbb{D}^* is the same as that for the process \mathbb{D} .

We call \mathbb{D}^* the *conditional diffusion process*. In the case that both of the end points are either regular and absorbing or exit, the result corresponding to Theorem 2.2 is already obtained by Ewens [1], [2, Sect. 4.6] and Karlin and Taylor [8, Ch. 15, Sect. 9]. We will prove Theorems 2.1 and 2.2 in the next section.

Example 2.3 Let $\mathbb{D} = [X(t) : t \geq 0, P_x : x \in [0, 1]]$ be the diffusion process whose generator is given by

$$L = \frac{1}{2} \frac{d^2}{dx^2} + \frac{\alpha - 1}{2x} \frac{d}{dx}, \quad 0 < x < 1,$$

where α is a real number. This process is referred to as the α -dimensional Bessel process on $[0, 1]$ if $\alpha > 0$. In particular, \mathbb{D} is the Brownian motion on $[0, 1]$ in the case that $\alpha = 1$.

The scale function s and the speed measure dm are given by

$$\begin{aligned} s(x) &= \begin{cases} \log x & \text{if } \alpha = 2, \\ \frac{1}{2 - \alpha} (x^{2 - \alpha} - 1) & \text{if } \alpha \neq 2, \end{cases} \\ m(x) &= \begin{cases} 2 \log x & \text{if } \alpha = 0, \\ \frac{2}{\alpha} (x^\alpha - 1) & \text{if } \alpha \neq 0. \end{cases} \end{aligned}$$

Consequently, the end point 1 is always regular. The end point 0 is exit if $\alpha \leq 0$, regular if $0 < \alpha < 2$, entrance if $\alpha \geq 2$. We assume that the end point 1 is absorbing since it is regular. The transition probability density $p(t, x, y)$ with respect to the speed measure dm is expressed in terms of the Bessel functions. Let $J_\nu(x)$ be the Bessel function defined by

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n! \Gamma(n + \nu + 1)}.$$

We set $\mathbb{J}(\nu) = \{x > 0 : J_\nu(x) = 0\}$. It is known that $\mathbb{J}(\nu)$ is a countably infinite set, it has no accumulating points in $[0, \infty)$, and

$$J_{\nu-1}(x) J_{\nu+1}(x) \neq 0 \quad \text{for } x \in \mathbb{J}(\nu),$$

(cf. [10, Ch. 15]). If the end point 0 is either exit or regular and absorbing, then

$$p(t, x, y) = \sum_{\kappa \in \mathbb{J}(1 - \frac{\alpha}{2})} \frac{1}{|J_{-\frac{\alpha}{2}}(\kappa) J_{2 - \frac{\alpha}{2}}(\kappa)|} e^{-\frac{1}{2} \kappa^2 t} (xy)^{1 - \frac{\alpha}{2}} J_{1 - \frac{\alpha}{2}}(\kappa x) J_{1 - \frac{\alpha}{2}}(\kappa y). \quad (2.14)$$

In the case that $\alpha = 1$, (2.14) is reduced to

$$p(t, x, y) = \sum_{n=1}^{\infty} e^{-\frac{1}{2}n^2\pi^2 t} \sin n\pi x \sin n\pi y. \quad (2.15)$$

This is exactly the transition probability density of the Brownian motion with absorbing end points 0 and 1. If the end point 0 is either entrance or regular and reflecting, then

$$p(t, x, y) = \sum_{\kappa \in \mathbb{J}(\frac{\alpha}{2}-1)} \frac{1}{|J_{\frac{\alpha}{2}-2}(\kappa)J_{\frac{\alpha}{2}}(\kappa)|} e^{-\frac{1}{2}\kappa^2 t} (xy)^{1-\frac{\alpha}{2}} J_{\frac{\alpha}{2}-1}(\kappa x) J_{\frac{\alpha}{2}-1}(\kappa y). \quad (2.16)$$

In the case that $\alpha = 1$, (2.16) is reduced to

$$p(t, x, y) = \sum_{n=1}^{\infty} e^{-\frac{1}{2}(\frac{2n-1}{2})^2\pi^2 t} \cos \frac{2n-1}{2}\pi x \cos \frac{2n-1}{2}\pi y. \quad (2.17)$$

This is exactly the transition probability density of the Brownian motion with reflecting end point 0 and absorbing end point 1.

Now we apply our results to this diffusion process. In the case that $\alpha \geq 2$, the conditional diffusion process \mathbb{D}^* is \mathbb{D} itself. In the case that $\alpha < 2$, the generator L^* is given by

$$L^* = \frac{1}{2} \frac{d^2}{dx^2} + \frac{3-\alpha}{2x} \frac{d}{dx}.$$

Since $\pi(x) = x^{2-\alpha}$ and $p_o(t, x, y)$ coincides with (2.14), the transition probability density $p^*(t, x, y)$ is given by

$$p^*(t, x, y) = \sum_{\kappa \in \mathbb{J}(1-\frac{\alpha}{2})} \frac{1}{|J_{-\frac{\alpha}{2}}(\kappa)J_{2-\frac{\alpha}{2}}(\kappa)|} e^{-\frac{1}{2}\kappa^2 t} (xy)^{\frac{\alpha}{2}-1} J_{1-\frac{\alpha}{2}}(\kappa x) J_{1-\frac{\alpha}{2}}(\kappa y).$$

3 Proof of main theorems

Let $\mathbb{D} = [X(t) : t \geq 0, P_x : x \in I]$ be the diffusion process given in the preceding section, that is, I is the interval with end points 0 and 1 and generator L is given by (2.1). Throughout this section we assume that (2.3) and (2.6) are satisfied.

Proof of Theorem 2.1 Let $\{\xi_n\}, \{\eta_n\}$ be sequences satisfying $\xi_n \downarrow 0$, $\eta_n \uparrow 1$ as $n \rightarrow \infty$, and \mathbb{D}_n ($n = 1, 2, \dots$) be the stopped diffusion

processes defined by $\mathbb{D}_n = [X(t \wedge \sigma_{\xi_n} \wedge \sigma_{\eta_n}) : t \geq 0, P_x : x \in [\xi_n, \eta_n]]$. We denote by $p_n(t, x, y)$ its transition probability density, that is,

$$P_x(X(t \wedge \sigma_{\xi_n} \wedge \sigma_{\eta_n}) \in E) = \int_E p_n(t, x, y) m_o(y) dy, \\ t > 0, x \in [\xi_n, \eta_n], E \in \mathcal{B}([\xi_n, \eta_n]). \quad (3.1)$$

Let $\mathbb{D}_o = [X_o(t) : t \geq 0, P_x : x \in (0, 1)]$ be the stopped diffusion process given in the preceding section and $p_o(t, x, y)$ be its transition probability density. Appealing to the convergence theorem of transition probability density due to Ogura [9, Proposition 5.1], we find that

$$\lim_{n \rightarrow \infty} p_n(t, x, y) = p_o(t, x, y) \quad \text{uniformly in } (t, x, y) \in K, \\ \text{for any compact sets } K \subset (0, \infty) \times (0, 1) \times (0, 1). \quad (3.2)$$

Let us fix $t \in (0, \infty)$, $x \in (0, 1)$ and $E \in \mathcal{B}((0, 1))$. By the definition

$$P_x(X(t) \in E | \sigma_{\eta_n} < \sigma_{\xi_n}) = \frac{P_x(X(t) \in E, \sigma_{\eta_n} < \sigma_{\xi_n})}{P_x(\sigma_{\eta_n} < \sigma_{\xi_n})}. \quad (3.3)$$

The numerator of the right-hand side of (3.3) is decomposed into two parts:

$$P_x(X(t) \in E, \sigma_{\eta_n} < \sigma_{\xi_n}) \\ = P_x(X(t) \in E, t < \sigma_{\xi_n} \wedge \sigma_{\eta_n}, \sigma_{\eta_n} < \sigma_{\xi_n}) \\ + P_x(X(t) \in E, t \geq \sigma_{\xi_n} \wedge \sigma_{\eta_n}, \sigma_{\eta_n} < \sigma_{\xi_n}). \quad (3.4)$$

First we evaluate the first term of the right-hand side of (3.4). Let us fix an $\varepsilon > 0$ arbitrarily. Then there exist two points a and b ($0 < a < x < b < 1$) such that

$$P_x(X(t) \in (0, a) \cup (b, 1)) < \varepsilon. \quad (3.5)$$

We may assume that $\xi_n < a$ and $b < \eta_n$ for sufficiently large n . We set $E_0 = E \cap ((0, a) \cup (b, 1))$ and $E_1 = E \cap [a, b]$. Then we find that

$$P_x(X(t) \in E, t < \sigma_{\eta_n} < \sigma_{\xi_n}) \\ = P_x(X(t) \in E_0, t < \sigma_{\eta_n} < \sigma_{\xi_n}) + P_x(X(t) \in E_1, t < \sigma_{\eta_n} < \sigma_{\xi_n}) \\ \equiv \text{I}_n + \text{II}_n,$$

$$\int_{E \cap (\xi_n, \eta_n)} p_n(t, x, y) \frac{s(y) - s(\xi_n)}{s(\eta_n) - s(\xi_n)} m_o(y) dy$$

$$\begin{aligned}
&= \int_{E_0 \cap (\xi_n, \eta_n)} p_n(t, x, y) \frac{s(y) - s(\xi_n)}{s(\eta_n) - s(\xi_n)} m_o(y) dy \\
&\quad + \int_{E_1} p_n(t, x, y) \frac{s(y) - s(\xi_n)}{s(\eta_n) - s(\xi_n)} m_o(y) dy \\
&\equiv \text{III}_n + \text{IV}_n, \\
&\quad \int_E p_o(t, x, y) \pi(y) m_o(y) dy \\
&= \int_{E_0} p_o(t, x, y) \pi(y) m_o(y) dy + \int_{E_1} p_o(t, x, y) \pi(y) m_o(y) dy \\
&\equiv \text{V} + \text{VI}.
\end{aligned}$$

By means of (3.5), we obtain that

$$\text{I}_n \leq P_x(X(t) \in (0, a) \cup (b, 1)) < \varepsilon, \quad (3.6)$$

$$\begin{aligned}
\text{III}_n &\leq \int_{E_0 \cap (\xi_n, \eta_n)} p_n(t, x, y) m_o(y) dy \\
&= P_x(X(t) \in E_0, t < \sigma_{\xi_n} \wedge \sigma_{\eta_n}) \\
&\leq P_x(X(t) \in (0, a) \cup (b, 1)) < \varepsilon,
\end{aligned} \quad (3.7)$$

$$\begin{aligned}
\text{V} &\leq \int_{E_0} p_o(t, x, y) m_o(y) dy \\
&\leq P_x(X_o(t) \in (0, a) \cup (b, 1)) \\
&\leq P_x(X(t) \in (0, a) \cup (b, 1)) < \varepsilon.
\end{aligned} \quad (3.8)$$

By using Markov property and (2.2), we obtain that

$$\begin{aligned}
\text{II}_n &= P_x(X(t) \in E_1, t < \sigma_{\xi_n} \wedge \sigma_{\eta_n}, \sigma_{\eta_n} < \sigma_{\xi_n}) \\
&= E_x[P_X(t)(\sigma_{\eta_n} < \sigma_{\xi_n}); X(t) \in E_1, t < \sigma_{\xi_n} \wedge \sigma_{\eta_n}] \\
&= \int_{E_1} p_n(t, x, y) P_y(\sigma_{\eta_n} < \sigma_{\xi_n}) m_o(y) dy \\
&= \int_{E_1} p_n(t, x, y) \frac{s(y) - s(\xi_n)}{s(\eta_n) - s(\xi_n)} m_o(y) dy = \text{IV}_n.
\end{aligned}$$

Since $p_o(t, x, y)$ is continuous in (t, x, y) and $\int_{E_1} m_o(y) dy < \infty$, we obtain that

$$\lim_{n \rightarrow \infty} \text{II}_n = \lim_{n \rightarrow \infty} \text{IV}_n = \int_{E_1} p_o(t, x, y) \pi(y) m_o(y) dy = \text{VI}, \quad (3.9)$$

by the bounded convergence theorem and (3.2). By (3.6), (3.7), (3.8) and (3.9), we have

$$\limsup_{n \rightarrow \infty} \left| P_x(X(t) \in E, t < \sigma_{\eta_n} < \sigma_{\xi_n}) - \int_{E \cap (\xi_n, \eta_n)} p_n(t, x, y) \frac{s(y) - s(\xi_n)}{s(\eta_n) - s(\xi_n)} m_o(y) dy \right| \leq 2\varepsilon, \quad (3.10)$$

$$\limsup_{n \rightarrow \infty} \left| \int_{E \cap (\xi_n, \eta_n)} p_n(t, x, y) \frac{s(y) - s(\xi_n)}{s(\eta_n) - s(\xi_n)} m_o(y) dy - \int_E p_o(t, x, y) \pi(y) m_o(y) dy \right| \leq 2\varepsilon. \quad (3.11)$$

The inequalities (3.10) and (3.11) lead us to

$$\lim_{n \rightarrow \infty} P_x(X(t) \in E, t < \sigma_{\eta_n} < \sigma_{\xi_n}) = \int_E p_o(t, x, y) \pi(y) m_o(y) dy, \quad \text{for } t \in (0, \infty), x \in (0, 1), E \in \mathcal{B}((0, 1)). \quad (3.12)$$

Next we evaluate the second term of the right-hand side of (3.4). If the end point 1 is either regular and absorbing or exit, then

$$\begin{aligned} & \lim_{n \rightarrow \infty} P_x(X(t) \in E, t \geq \sigma_{\xi_n} \wedge \sigma_{\eta_n}, \sigma_{\eta_n} < \sigma_{\xi_n}) \\ & \leq \lim_{n \rightarrow \infty} P_x(X(t) \in E, t \geq \sigma_{\eta_n}) = 0. \end{aligned} \quad (3.13)$$

Let us assume that the end point 1 is natural. Then it is known that

$$E_x[e^{-\alpha \sigma_{\eta_n}}] = \frac{g(x, \alpha)}{g(\eta_n, \alpha)}, \quad \lim_{n \rightarrow \infty} g(\eta_n, \alpha) = \infty, \quad \alpha > 0,$$

where g is a positive increasing solution of $Lg = \alpha g$ (cf. [7, Sect. 4.6]). Therefore we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} P_x(X(t) \in E, t \geq \sigma_{\xi_n} \wedge \sigma_{\eta_n}, \sigma_{\eta_n} < \sigma_{\xi_n}) \\ & \leq \lim_{n \rightarrow \infty} P_x(X(t) \in E, t \geq \sigma_{\eta_n}) \\ & \leq e \lim_{n \rightarrow \infty} E_x[e^{-\frac{\sigma_{\eta_n}}{t}}] \\ & = e \lim_{n \rightarrow \infty} \frac{g(x, \alpha)}{g(\eta_n, \alpha)} = 0. \end{aligned} \quad (3.14)$$

By means of (2.4), (3.3), (3.12), (3.13), (3.14), we obtain that

$$\lim_{n \rightarrow \infty} P_x(X(t) \in E | \sigma_{\eta_n} < \sigma_{\xi_n}) = \int_E p_o(t, x, y) \frac{\pi(y)}{\pi(x)} m_o(y) dy.$$

Thus Theorem 2.1 is proved.

Proof of Theorem 2.2 (i) Since $p^*(t, x, y)$ is symmetric in x and y , we only prove $\frac{\partial}{\partial t} p^*(t, x, y) = L_x^* p^*(t, x, y)$. We note that $\frac{\partial}{\partial t} p_o(t, x, y) = L_x p_o(t, x, y)$. In the case that $s(0+) = -\infty$, (2.13) is trivial because $\pi(x) = 1$ for $x \in (0, 1)$, $p^*(t, x, y) = p_o(t, x, y)$ for $t > 0$, $x, y \in (0, 1)$, and $L^* = L$. In the case that $s(0+) > -\infty$, we obtain the following:

$$\begin{aligned}
& \frac{1}{2} a^*(x) \frac{\partial^2}{\partial x^2} p^*(t, x, y) + b^*(x) \frac{\partial}{\partial x} p^*(t, x, y) \\
= & \frac{1}{2} a(x) \frac{\partial^2}{\partial x^2} \frac{p_o(t, x, y)}{\pi(x)\pi(y)} + \left\{ b(x) + \frac{s_o(x)}{s(x) - s(0+)} a(x) \right\} \frac{\partial}{\partial x} \frac{p_o(t, x, y)}{\pi(x)\pi(y)} \\
= & \frac{1}{2} \frac{a(x)}{\pi(x)\pi(y)} \frac{\partial^2}{\partial x^2} p_o(t, x, y) + \frac{b(x)}{\pi(x)\pi(y)} \frac{\partial}{\partial x} p_o(t, x, y) \\
= & \frac{1}{\pi(x)\pi(y)} \frac{\partial}{\partial t} p_o(t, x, y) \\
= & \frac{\partial}{\partial t} p^*(t, x, y).
\end{aligned}$$

Thus we obtain (2.13).

Proof of Theorem 2.2 (ii) Combining (2.11) with the first statement of the theorem, we see that $P^*(t, x, E)$ is the transition probability of a diffusion process \mathbb{D}^* whose generator is given by L^* .

Proof of Theorem 2.2 (iii) Assume that $s(0+) = -\infty$. Then the end point 0 is entrance or natural, $\pi(x) = 1$ for every $x \in (0, 1)$, $s_o^* = s_o$ and $m_o^* = m_o$. Therefore the end point 0 is the entrance or natural for \mathbb{D}^* according as the end point 0 is entrance or natural for \mathbb{D} . At the same time we also see that the end point 1 is regular, exit or natural for \mathbb{D}^* according as the end point 1 is regular, exit or natural for \mathbb{D} .

Assume that $s(0+) > -\infty$. Then it follows that

$$\begin{aligned}
s^*(0+) &= - \int_0^c s_o^*(x) dx = - \int_0^c \frac{s_o(x)}{\pi(x)^2} dx \\
&= -(s(1-) - s(0+))^2 \int_0^c \frac{s_o(x)}{(s(x) - s(0+))^2} dx = -\infty.
\end{aligned}$$

Therefore the end point 0 is either entrance or natural for \mathbb{D}^* . So we

have to see the quantity $J^*(0+) = \int_c^0 m_o^*(x) dx \int_c^x s_o^*(y) dy$. We have

$$J^*(0+) = \int_0^c (s(x) - s(0+))^2 m_o(x) dx \int_x^c \frac{s_o(y)}{(s(y) - s(0+))^2} dy$$

$$= \int_0^c (s(x) - s(0+)) \frac{s(c) - s(x)}{s(c) - s(0+)} m_o(x) dx.$$

Since $s(0+) > -\infty$, the end point 0 is regular, exit or natural for \mathbb{D} . If the end point 0 is regular or exit for \mathbb{D} , then

$$J^*(0+) \leq \int_0^c (s(x) - s(0+)) m_o(x) dx = I(0+) < \infty,$$

which implies that the end point 0 is entrance for \mathbb{D}^* . If the end point 0 is natural for \mathbb{D} , then, for any d satisfying $0 < d < c$,

$$\begin{aligned} J^*(0+) &> \int_0^d (s(x) - s(0+)) \frac{s(c) - s(d)}{s(c) - s(0+)} m_o(x) dx \\ &= \frac{s(c) - s(d)}{s(c) - s(0+)} \int_0^d s_o(y) dy \int_y^d m_o(x) dx = \infty, \end{aligned}$$

which implies that the end point 0 is natural for \mathbb{D}^* . By means of (2.10),

$$s^*(1-) = \int_c^1 s_o^*(x) dx = \int_c^1 \frac{s_o(x)}{\pi(x)^2} dx < \infty.$$

Therefore the end point 1 is regular, exit or natural for \mathbb{D}^* . Since $s(0+) > -\infty$, we have

$$\begin{aligned} J^*(1-) &= \int_c^1 m_o^*(x) dx \int_c^x s_o^*(y) dy \\ &= \int_c^1 (s(x) - s(0+))^2 m_o(x) dx \int_c^x \frac{s_o(y)}{(s(y) - s(0+))^2} dy \\ &= \int_c^1 (s(x) - s(0+)) \frac{s(x) - s(c)}{s(c) - s(0+)} m_o(x) dx, \\ I^*(1-) &= \int_c^1 s_o^*(x) dx \int_c^x m_o^*(y) dy \\ &= \int_c^1 (s(y) - s(0+)) \frac{s(1-) - s(y)}{s(1-) - s(0+)} m_o(y) dy. \end{aligned}$$

If the end point 1 is regular for \mathbb{D} , then

$$\begin{aligned} J^*(1-) &< \frac{(s(1-) - s(0+))(s(1-) - s(c))}{s(c) - s(0+)} \int_c^1 m_o(x) dx < \infty, \\ I^*(1-) &< (s(1-) - s(c)) \int_c^1 m_o(x) dx < \infty. \end{aligned}$$

Therefore the end point 1 is regular for \mathbb{D}^* . If the end point 1 is exit, then

$$\begin{aligned}
 J^*(1-) &> \int_c^1 (s(x) - s(c))m_o(x)dx \\
 &= \int_c^1 m_o(x)dx \int_c^x s_o(y)dy = J(1-) = \infty, \\
 I^*(1-) &< \int_c^1 (s(1-) - s(y))m_o(y)dy \\
 &= \int_c^1 m_o(y)dy \int_y^1 s_o(x)dx = I(1-) < \infty.
 \end{aligned}$$

Therefore the end point 1 is exit for \mathbb{D}^* . If the end point 1 is natural for \mathbb{D} , then

$$\begin{aligned}
 J^*(1-) &> \int_c^1 (s(x) - s(c))m_o(x)dx \\
 &= \int_c^1 m_o(x)dx \int_c^x s_o(y)dy = J(1-) = \infty, \\
 I^*(1-) &> \frac{s(c) - s(0+)}{s(1-) - s(0+)} \int_c^1 (s(1-) - s(y))m_o(y)dy \\
 &= \frac{s(c) - s(0+)}{s(1-) - s(0+)} \int_c^1 m_o(y)dy \int_y^1 s_o(x)dx = I(1-) = \infty.
 \end{aligned}$$

Therefore the end point 1 is natural for \mathbb{D}^* . We thus obtain the third statement of the theorem.

4 Application to population genetics

We consider a locus with two alleles in a randomly mating population of N diploid individuals. We denote by A_1 the mutant allele and by A_2 the wild-type allele. Let $X(n)$ be the relative frequency of A_1 at the n -th generation in the population ($n = 0, 1, 2, \dots$). Mutation, selection and random sampling drift are the factors which change gene frequency $X(n)$. The Wright-Fisher model and the stochastic selection model are the fundamental stochastic models in population genetics. The Wright-Fisher model is the stochastic model due to random sampling drift and this stochastic force has no correlation between distinct generations. On the other hand in a stochastic selection model, stochastic force of selection has autocorrelation from generation to generation in general. These

models are described by discrete time stochastic processes because we regard the generation as the time unit. It is difficult, however, to analyze these discrete time models. Then diffusion approximations are employed for the original discrete time models. In other words, we approximate a discrete time stochastic process in population genetics by an appropriate diffusion process by introducing a new time scaling. For approximating methods and applicability of diffusion approximations, see [4], [6] and cited therein. A general stochastic model may be obtained by combining two diffusion models that correspond to the Wright-Fisher model and the stochastic selection model. The obtained diffusion model becomes more realistic by adding two stochastic factors. We will deal with a diffusion process $\mathbb{D} = [X(t) : t \geq 0, P_x : x \in I]$ that corresponds to the diffusion model with random sampling drift and stochastic selection, where I is the interval with end points 0 and 1. Further we introduce two deterministic factors of mutation and deterministic selection in this diffusion model.

It is known that the generator of the diffusion process \mathbb{D} is given by

$$\begin{aligned} L &= \frac{1}{2}a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}, \\ a(x) &= \frac{1}{2N}x(1-x) + \gamma x^2(1-x)^2, \\ b(x) &= v - (u+v)x + \rho\frac{\gamma}{2}x(1-x)(1-2x) + \alpha x(1-x)(h - (2h-1)x). \end{aligned} \tag{4.1}$$

The meaning and domain of each variable are as follows. The variable x is the relative frequency of A_1 ($0 \leq x \leq 1$). The parameter N is the population size ($2 \leq N \leq \infty$). Note that the case that $N = \infty$ corresponds to that without random sampling drift. Three genotypes A_1A_1 , A_1A_2 and A_2A_2 have fitnesses $1+s_n+\alpha$, $1+\frac{1}{2}s_n+h\alpha$ and 1 where s_n is the stochastic part of selection coefficient at the n -th generation, α is the deterministic part of selection coefficient ($s_n + \alpha \geq -1$) and h is a parameter showing the dominance of A_1 ($0 \leq h \leq 1$). It is assumed that stochastic selection has no dominance. We assume that $\{s_n : 0, \pm 1, \pm 2, \dots\}$ is a discrete time

stationary process with $E[s_n] = 0$. The parameter $\gamma = \frac{1}{4} \sum_{k=-\infty}^{\infty} E[s_0 s_k]$

is a degree of autocorrelated stochastic selection ($0 \leq \gamma < \infty$). The parameter ρ denotes the type of stochastic selection ($\rho \geq 1$). The case that $\rho = 1$ with $N < \infty$ is called the TIM model and the case that $\rho > 1$ is called the SAS-CFF model [4]. The mutation rate per generation from A_1 to A_2 [resp. from A_2 to A_1] is denoted by u [resp. v] ($0 \leq u, v \leq 1$). We assume that $\alpha \cdot \gamma = 0$ because we consider that at most stochastic

selection or deterministic selection exists in the following for simplicity. Moreover $N < \infty$ or $\gamma > 0$ is satisfied because a diffusion coefficient must be positive. The state of the end points is described in Tables 1 and 2 (cf. [5]).

Table 1

	$s(0+)$	$m(0+)$	state
$N < \infty, v = 0$	$> -\infty$	$= -\infty$	exit
$N < \infty, 0 < 4Nv < 1$	$> -\infty$	$> -\infty$	regular
$N < \infty, 4Nv \geq 1$	$= -\infty$	$> -\infty$	entrance
$N = \infty, v > 0$	$= -\infty$	$> -\infty$	entrance
$N = \infty, v = 0, u < \frac{\gamma}{2}(\rho - 1)$	$= -\infty$	$> -\infty$	natural
$N = \infty, v = 0, u = \frac{\gamma}{2}(\rho - 1)$	$= -\infty$	$= -\infty$	natural
$N = \infty, v = 0, u > \frac{\gamma}{2}(\rho - 1)$	$> -\infty$	$= -\infty$	natural

Table 2

	$s(1-)$	$m(1-)$	state
$N < \infty, u = 0$	$< \infty$	$= \infty$	exit
$N < \infty, 0 < 4Nu < 1$	$< \infty$	$< \infty$	regular
$N < \infty, 4Nu \geq 1$	$= \infty$	$< \infty$	entrance
$N = \infty, u > 0$	$= \infty$	$< \infty$	entrance
$N = \infty, u = 0, v < \frac{\gamma}{2}(\rho - 1)$	$= \infty$	$< \infty$	natural
$N = \infty, u = 0, v = \frac{\gamma}{2}(\rho - 1)$	$= \infty$	$= \infty$	natural
$N = \infty, u = 0, v > \frac{\gamma}{2}(\rho - 1)$	$< \infty$	$= \infty$	natural

Now we apply results of the previous sections. It follows from Table 2 that (2.3) and (2.6) are satisfied if one of the following (4.2), (4.3), (4.4) is satisfied.

$$N < \infty, u = 0. \quad (4.2)$$

$$N < \infty, 0 < 4Nu < 1. \quad (4.3)$$

In this case the end point 1 is regular and hence it is assumed to be absorbing. This assumption means that the gene frequency process is stopped at the first hitting time to 1 though it may leave this

point.

$$N = \infty, \quad u = 0, \quad v > \frac{\gamma}{2}(\rho - 1). \quad (4.4)$$

We assume that one of (4.2), (4.3) and (4.4) is satisfied in the following. By means of Theorem 2.1, there exists the limit of the following conditional distribution:

$$P^*(t, x, E) = \lim_{n \rightarrow \infty} P_x(X(t) \in E | \sigma_{\eta_n} < \sigma_{\xi_n}), \quad t \geq 0, \quad x \in (0, 1), \quad E \in \mathcal{B}((0, 1)), \quad (4.5)$$

for any sequences $\{\xi_n\}$, $\{\eta_n\}$ satisfying $\xi_n \downarrow 0$ and $\eta_n \uparrow 1$ as $n \rightarrow \infty$. The limit $P^*(t, x, y)$ is independent of choice of sequences $\{\xi_n\}$ and $\{\eta_n\}$. The function $P^*(t, x, \cdot)$ is absolutely continuous and it holds that

$$P^*(t, x, E) = \int_E p_o(t, x, y) \frac{\pi(y)}{\pi(x)} m_o(y) dy, \quad t > 0, \quad x \in (0, 1), \quad E \in \mathcal{B}((0, 1)). \quad (4.6)$$

First, we consider the case that $N < \infty$. Assume that (4.2) or (4.3) is satisfied. If $0 \leq 4Nv < 1$, then $s(0+) > -\infty$ by Table 1, and hence the generator L^* of the conditional diffusion process \mathbb{D}^* is given by

$$L^* = \frac{1}{2} a^*(x) \frac{d^2}{dx^2} + b^*(x) \frac{d}{dx} = \frac{d}{dm^*} \frac{d}{ds^*}, \quad (4.7)$$

where

$$\begin{aligned} a^*(x) &= a(x), \quad b^*(x) = b(x) + \frac{s_o(x)}{s(x) - s(0+)} a(x), \\ s_o^*(x) &= \exp \left\{ -2 \int_c^x \frac{b^*(y)}{a^*(y)} dy \right\} = \frac{s_o(x)}{\pi(x)^2}, \\ m_o^*(x) &= \frac{2}{a^*(x)} \exp \left\{ 2 \int_c^x \frac{b^*(y)}{a^*(y)} dy \right\} = \pi(x)^2 m_o(x), \\ s^*(x) &= \int_c^x s_o^*(y) dy, \quad m^*(x) = \int_c^x m_o^*(y) dy. \end{aligned}$$

The explicit form $b^*(x)$ can be written as

$$\begin{aligned} b^*(x) &= v - (u + v)x + \alpha x(1 - x)(h - (2h - 1)x) \\ &\quad + \frac{x^{1-4Nv}(1 - x)^{1-4Nu} e^{-4N\alpha hx + 2N\alpha(2h-1)x^2}}{2N \int_0^x y^{-4Nv}(1 - y)^{-4Nu} e^{-4N\alpha hy + 2N\alpha(2h-1)y^2} dy}. \end{aligned}$$

If $4Nv \geq 1$, then $s(0+) = -\infty$. Hence the conditional diffusion process \mathbb{D}^* is no other than \mathbb{D} . We can obtain a more detailed result if $\alpha = \gamma = 0$.

Example 4.1 (No selection model) We consider the case that $N < \infty$, $0 \leq 4Nu < 1$, $0 \leq 4Nv < 1$ and $\alpha = \gamma = 0$. The generator is given by

$$L = \frac{1}{4N}x(1-x)\frac{d^2}{dx^2} + \{v - (u+v)x\}\frac{d}{dx}. \quad (4.8)$$

By Table 1, the end point 0 is either regular or exit. By Table 2, the end point 1 is either regular or exit. We assume that the end point 1 is absorbing if it is regular. It is easy to see that

$$\pi(x) = P_x(\sigma_1 < \sigma_0) = \frac{\int_0^x y^{-4Nv}(1-y)^{-4Nu}dy}{\int_0^1 y^{-4Nv}(1-y)^{-4Nu}dy},$$

which is the probability that A_1 is fixed before it disappears. By means of (4.2) and (4.3), (2.3) and (2.6) are satisfied. Therefore

$$\begin{aligned} P_x^*(X^*(t) \in E) &= P_x(X(t) \in E | \sigma_1 < \sigma_0) \\ &= 4N \int_E p_o(t, x, y) \frac{\int_0^y z^{-4Nv}(1-z)^{4Nu}dz}{\int_0^x z^{-4Nv}(1-z)^{4Nu}dz} y^{4Nv-1}(1-y)^{4Nu-1}dy, \\ &\quad \text{for } t > 0, x \in (0, 1), E \in \mathcal{B}((0, 1)), \end{aligned} \quad (4.9)$$

where

$$\begin{aligned} p_o(t, x, y) &= \frac{1}{4N [\Gamma(2-4Nv)]^2} x^{1-4Nv}(1-x)^{1-4Nu} y^{1-4Nv}(1-y)^{1-4Nu} \\ &\quad \times \sum_{i=1}^{\infty} F(2-4N(u+v)+i, 1-i, 2-4Nv, x) \\ &\quad \times F(2-4N(u+v)+i, 1-i, 2-4Nv, y) \\ &\quad \times \frac{(1-4N(u+v)+2i) \Gamma(2-4N(u+v)+i) \Gamma(1-4Nv+i)}{(i-1)! \Gamma(i+1-4Nu)} \\ &\quad \times e^{-\frac{1}{4N}(i^2+(1-4N(u+v))i)t}. \end{aligned}$$

Here $F(\alpha, \beta, \gamma, x)$ is the hypergeometric function defined by

$$F(\alpha, \beta, \gamma, x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)} \frac{x^n}{n!}.$$

Note that (4.9) with $u = v = 0$ has been obtained by Ewens [1].

Next, we consider the case that $N = \infty$.

Example 4.2 (Stochastic selection model without random sampling drift)

We consider the case that $N = \infty$, $\gamma > 0$, $v > \frac{\gamma}{2}(\rho - 1)$, $u = \alpha = 0$.

The generator is given by

$$L = \gamma x^2(1-x)^2 \frac{d^2}{dx^2} + \left\{ v(1-x) + \frac{1}{2}\gamma\rho x(1-x)(1-2x) \right\} \frac{d}{dx}. \quad (4.10)$$

The density of scale function is given by $s_o(x) = x^{-\frac{2v}{\gamma}-\rho}(1-x)^{\frac{2v}{\gamma}-\rho}e^{\frac{2v}{\gamma}\frac{1}{x}}$ and that of speed measure is given by $m_o(x) = \frac{2}{\gamma x^2(1-x)^2}x^{\frac{2v}{\gamma}+\rho}(1-x)^{-\frac{2v}{\gamma}+\rho}e^{-\frac{2v}{\gamma}\frac{1}{x}}$. By Table 1, the end point 0 is entrance. By Table 2, the end point 1 is natural. Since $s(0+) = -\infty$, it holds that $\pi(x) = 1$ for $x \in (0, 1)$. By means of (4.4), (2.3) and (2.6) are satisfied. Applying our results, we obtain that

$$\begin{aligned} P_x^*(X^*(t) \in E) &= P_x(X(t) \in E | \sigma_1 < \sigma_0) = P_x(X(t) \in E) \\ &= \frac{2}{\gamma} \int_E p(t, x, y) y^{\frac{2v}{\gamma}+\rho-2} (1-y)^{-\frac{2v}{\gamma}+\rho-2} e^{-\frac{2v}{\gamma}\frac{1}{y}} dy, \\ &\quad \text{for } t > 0, x \in (0, 1), E \in \mathcal{B}((0, 1)), \end{aligned} \quad (4.11)$$

where $p(t, x, y)$ is the probability density function of \mathbb{D} .

The above results mean that for a diffusion model in population genetics defined by the generator (4.1), the conditional diffusion model $X^*(t)$ is different from the original diffusion model $X(t)$ only if $N < \infty$, $0 \leq 4Nu < 1$ and $0 \leq 4Nv < 1$. If one of the these inequalities does not holds, then the conditional diffusion model is the same as the original diffusion model or it is not defined since at least one of the end points 0 and 1 is not accessible. By considering two points ε and $1 - \varepsilon$ instead of 0 and 1 ($0 < \varepsilon < 1$), the conditional diffusion model seems to be different from the original diffusion model. It is a future problem to investigate such a conditional diffusion process.

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Conditional diffusion models

MAENO Miyuki

We consider a one-dimensional diffusion process conditioned by hitting times. We call this process a conditional diffusion process. We obtain the generator of the conditional diffusion process. We compare the original generator with that of the conditional diffusion process, and find a relation between these generators. By using this relation, we show some properties of the conditional diffusion process.

Our results are applied to Bessel processes and diffusion models in population genetics. In population genetics, the change of gene frequency is approximated by a one-dimensional diffusion process on $[0,1]$. We consider the diffusion model with random sampling drift and stochastic selection as the stochastic factors of gene frequency change. For the case that the gene in question is surely lost from the population or fixed in the population, some properties of the conditional diffusion processes in population genetics have been studied. The end points are accessible in this case. Our results are applicable, however, to the case that the end points are not necessary accessible.